D-INFK
Afonso Bandeira
Bernd Gärtner

Linear Algebra

HS 2023

Assignment 3

Submission Deadline: **17 October, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Elimination, back substitution, LU factorization (hand-in) (★☆☆)

a) Compute an LU factorization of the matrix $A = \begin{bmatrix} 2 & -12 & 6 \\ 1 & -4 & 1 \\ 2 & -11 & -5 \end{bmatrix}$.

b) Given the factorization $A = LU$ from above, solve the linear system $L\mathbf{y} = \mathbf{b}$ with $\mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 25 \end{bmatrix}$.

c) Given the solution $\mathbf{y}$ to the linear system $L\mathbf{y} = \mathbf{b}$ above, solve the linear system $U\mathbf{x} = \mathbf{y}$ (for $\mathbf{x}$).

d) Given the solution $\mathbf{x}$ for the system above, prove that $A\mathbf{x} = \mathbf{b}$, i.e. $\mathbf{x}$ also solves this system.

2. Matrix inverse (★☆☆)

a) Consider the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and the standard unit vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ for the three systems $A\mathbf{x}_1 = \mathbf{e}_1$, $A\mathbf{x}_2 = \mathbf{e}_2$, $A\mathbf{x}_3 = \mathbf{e}_3$.

b) What is the inverse A^{-1} of A?

c) What is the inverse D^{-1} of the diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$?

d) What is the inverse B^{-1} of the matrix $B = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$?

3. Matrix inverse (★★☆)

a) Let A be an $n \times n$ matrix with inverse A^{-1} and let $k \in \mathbb{N}^+$ be an arbitrary integer. Does A^k have an inverse and if yes, what is it?

b) Recall the definition of a nilpotent matrix from Assignment 2: We say that a square matrix A is nilpotent if and only if there exists $k \in \mathbb{N}$ such that $A^k = 0$. Prove that a nilpotent matrix A cannot have an inverse.

c) Let A be an $n \times n$ matrix with $A^3 = I$ and $A^4 = I$. Prove that $A = I$.

d) Find a 2×2 matrix $A \neq I$ such that $A^k = I$ for all even k and $A^k = A$ for all odd $k \in \mathbb{N}$.

e) Can you also find a 2×2 matrix A that, for all $k \in \mathbb{N}$, satisfies $A^k = I$ if and only if $k \equiv_4 0$ (i.e. $A^k = I$ if and only if k is a multiple of 4)?

4. Hyperplanes (★★☆)

Consider the following set $S = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{d} = c\}$ for some vector $\mathbf{d} \in \mathbb{R}^n$ and some non-zero constant $c \in \mathbb{R}$. Notice that S is not a hyperplane according to the definition from the lecture because c is not zero. Now consider the set

$$S' = \{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1} : \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in S\}$$

which is a subset of $\mathbb{R}^{n+1}$. Prove that S' is a subset of a hyperplane of $\mathbb{R}^{n+1}$.

5. Commutativity of the inverse (★★★)

This is an exercise from page 17 of the blackboard notes: Given two $n \times n$ matrices A and B with $AB = I$, prove that $BA = I$. You can either try to solve this directly or proceed according to the following subtasks which should guide you through the proof.

a) Prove that the columns of B are linearly independent, i.e. B has rank n.

b) Use a) to prove that the columns of A are also linearly independent.

c) Use b) to prove that $BA - I = 0$.