D-INFK
Afonso Bandeira
Bernd Gärtner

Linear Algebra

HS 2023

Assignment 4

Submission Deadline: **24 October, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Subspaces of vector spaces (hand-in) (★★☆)

a) Let H be a hyperplane of $\mathbb{R}^n$. Prove that H is a subspace of $\mathbb{R}^n$.

b) In this exercise we consider the vector space V of all real-valued function over $\mathbb{R}$ on the interval $[0, 1]$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : [0, 1] \to \mathbb{R}$ and conversely, every function $\mathbf{f} : [0, 1] \to \mathbb{R}$ is in V. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

$$(\mathbf{f}+\mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in [0,1]$$
$$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in [0,1] \text{ and } c \in \mathbb{R}.$$

Prove that

$$U = \{\mathbf{f} \in V : \mathbf{f}(x) = \mathbf{f}(1-x) \text{ for all } x \in [0,1]\} \subseteq V$$

is a subspace of V.

2. Invertibility (★☆☆)

Let $A \in \mathbb{R}^{3\times 3}$ be the following upper triangular matrix with $a, b, c, d \in \mathbb{R}$:

$$A = \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}.$$

For which values of a, b, c, d is A invertible? Specify A^{-1} for these cases.

3. Transpose (★☆☆)

a) Find two 2×2 matrices A and B such that $(AB)^\top \neq A^\top B^\top$.

b) Can you also find two symmetric 2×2 matrices A, B with $(AB)^\top \neq A^\top B^\top$?

4. LU-Decomposition (★☆☆)

a) Determine an LU decomposition $PA = LU$ of the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 \\ -4 & -8 & 13 \\ 2 & -5 & -3 \end{bmatrix}.$$

b) For a different matrix B, we are given the LU decomposition $PB = LU$ with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solve the system of equations $B\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix}.$$

5. Product of triangular matrices (★★☆)

a) Let A and B be $n \times n$ lower triangular matrices. Prove that AB is lower triangular.

b) Let A and B be $n \times n$ upper triangular matrices. Prove that AB is upper triangular.

Hint: *Use the statement from subtask a).*

6. Inverse of triangular matrices (★★★)

a) Find the inverse of the 2×2 matrix $L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ where $a \in \mathbb{R}$.

b) Prove that a square lower triangular matrix is invertible if and only if all its diagonal entries are non-zero.

c) Prove that the inverse of any lower triangular matrix, if it exists, is lower triangular itself.

d) Are the statements of b) and c) also true if we replace *lower triangular* by *upper triangular*?

7. Subspaces (★☆☆)

1. Let U_1, U_2 be subspaces of $\mathbb{R}^n$. Which of the following subsets of $\mathbb{R}^n$ are also subspaces of $\mathbb{R}^n$?

(a) $U_1 \cap U_2$

(b) $U_1 \cup U_2$

(c) $U_1 \setminus U_2 := \{\mathbf{u} \in U_1 : \mathbf{u} \notin U_2\}$

(d) $\varnothing$

(e) $\{0\}$

(f) $U_1 + U_2 := \{\mathbf{u}_1 + \mathbf{u}_2 : \mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2\}$