D-INFK Linear Algebra HS 2023
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Assignment 5

Submission Deadline: **31 October, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Nullspace and column space (hand-in) (★★☆)

Let $\mathbf{v}$ be a *unit vector* (i.e. $\|\mathbf{v}\| = 1$) in $\mathbb{R}^3$. Consider the 3×3 matrices A and P defined by

$$A := \mathbf{v}\mathbf{v}^\top, \qquad P := I_3 - \mathbf{v}\mathbf{v}^\top = I_3 - A$$

where I_3 is the 3×3 identity matrix.

a) Calculate A^2 and P^2. Try to simplify the expressions you get as much as possible.

b) Let $\mathbf{w} \in \mathbb{R}^3$ be orthogonal to $\mathbf{v}$ (i.e. $\mathbf{w} \cdot \mathbf{v} = 0$). Prove $A\mathbf{w} = \mathbf{0}$.

c) Now let $\mathbf{w} \in \mathbb{R}^3$ be a vector satisfying $A\mathbf{w} = \mathbf{0}$. Prove $\mathbf{w} \cdot \mathbf{v} = 0$.

d) Based on b) and c), describe the nullspace $\mathbf{N}(A)$.

e) Determine the rank of A. Is A invertible?

f) Prove that $\mathbf{C}(A) = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$.

g) Also prove that $\mathbf{C}(A) = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$.

h) Use g) to prove $\mathbf{N}(P) = \mathbf{C}(A)$.

i) Finally, prove $\mathbf{C}(P) = \mathbf{N}(A)$.

Hint: In every subtask you may of course use statements that you have already proven in previous subtasks. For some of the subtasks we specifically tell you which previous subtasks might be helpful.

2. A subspace of $\mathbb{R}^n$ (★☆☆)

Note that Section 3.3 of the blackboard notes is a prerequisite for this exercise. We recommend solving it after the lecture on Friday (27.10.23).

Consider the vector subspace $U = \text{span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \mathbb{R}^4$ and the vector $\mathbf{b} \in \mathbb{R}^4$ with

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -4 \\ 8 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 5 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ -5 \\ 5 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 6 \\ 2 \end{bmatrix}.$$

a) Is $\mathbf{b}$ an element of U?

b) Are the three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ a basis of U?

3. Interpolation (★☆☆) Assume that you gathered the following datapoints

x	y
0	1
2	2
4	5
6	6

and you want to find a function $f : \mathbb{R} \to \mathbb{R}$ that interpolates them, i.e. f should satisfy $f(x) = y$ for all pairs of x, y given by the table above. There is an abundance of functions that you can try and in particular, there are many different functions that do interpolate the four datapoints. In this exercise, we are interested in polynomials, i.e. we restrict f to be a polynomial of degree at most 3. In particular, this means that f has the form $f(x) = ax^3 + bx^2 + cx + d$ for some $a, b, c, d \in \mathbb{R}$. Your task is to find values for a, b, c, d such that f interpolates all four points given in the table.

4. Subspaces (★★★)

In this exercise we consider the vector space V of all real-valued function on $\mathbb{R}$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ and conversely, every function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is in V. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

$$(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in \mathbb{R}$$
$$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in \mathbb{R} \text{ and } c \in \mathbb{R}.$$

Now consider the set of odd functions

$$O = \{\mathbf{f} \in V : \mathbf{f}(-x) = -\mathbf{f}(x) \text{ for all } x \in \mathbb{R}\}$$

and the set of even functions

$$E = \{\mathbf{f} \in V : \mathbf{f}(-x) = \mathbf{f}(x) \text{ for all } x \in \mathbb{R}\}.$$

a) Prove that both O and E are subspaces of V.

b) Prove that the intersection $O \cap E$ contains only the zero function $\mathbf{0} : \mathbb{R} \to \mathbb{R}$ with $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

c) Prove that any function $\mathbf{f} \in V$ can be written as $\mathbf{f} = \mathbf{g} + \mathbf{h}$ for some $\mathbf{g} \in E$ and $\mathbf{h} \in O$.

5. 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

Which of the following sets of vectors is a basis of $\mathbb{R}^4$?

(a)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(c)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

2. Which of the following matrices are in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$