D-INFK
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Linear Algebra

HS 2023

Assignment 6

Submission Deadline: **07 November, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. Underdetermined linear system (hand-in) (★☆☆)

Consider the underdetermined linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} -1 & 2 & 5 & -2 \\ -3 & 3 & 12 & -3 \\ 1 & -14 & -7 & -6 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} -6 \\ -15 \\ 8 \end{pmatrix}.$$

a) Determine the set of solutions $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{b}\}$, i.e. write down an *explicit* characterization of this set of solutions.

Hint: In the lecture you learned that every solution can be obtained from a particular solution and a basis of $\mathbf{N}(A)$. Hence, an explicit characterization of $\mathcal{L}$ can be given by finding such a particular solution and a basis of $\mathbf{N}(A)$ and then describing the possible combinations that are solutions.

b) Write down a basis for $\mathbf{N}(A)$ (you might have already found it in the previous subtask), and also find a basis for $\mathbf{C}(A)$.

c) What are the dimensions of $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$, and $\mathbf{C}(A^\top)$?

d) Determine a basis of $\mathbf{C}(A^\top)$.

2. Reconstruct a matrix (★☆☆)

Let A be a 3×2 matrix satisfying

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \text{ and } A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}.$$

a) Determine A.

b) Determine the dimensions of the four fundamental subspaces $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$, $\mathbf{C}(A^\top)$ of A.

3. Row operations preserve row space (★★☆)

In this exercise we prove that row operations preserve the row space of a matrix. This is an exercise from Section 3.5 of the blackboard notes.

Consider an $m \times n$ matrix A with rows $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ and an $m \times n$ matrix B with rows $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n \in \mathbb{R}^n$, i.e.

$$A = \begin{bmatrix} — & \mathbf{v}_1^\top & — \\ — & \mathbf{v}_2^\top & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{v}_m^\top & — \end{bmatrix} \text{ and } B = \begin{bmatrix} — & \mathbf{w}_1^\top & — \\ — & \mathbf{w}_2^\top & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{w}_m^\top & — \end{bmatrix}.$$

For each of the following tasks, let $i, j \in [m]$ with $i \neq j$ and $c \in \mathbb{R}$ be arbitrary.

a) Assume first that B was obtained from A by subtracting c times row i from row j of A. Prove that then $\mathbf{R}(A) = \mathbf{R}(B)$.

Hint: Concretely, we have $\mathbf{w}_k = \mathbf{v}_k$ for all $k \in [m] \setminus \{j\}$ and $\mathbf{w}_j = \mathbf{v}_j - c\mathbf{v}_i$.

b) Assume now instead that B was obtained from A by switching rows i and j. Prove that then $\mathbf{R}(A) = \mathbf{R}(B)$ holds as well.

c) Finally, assume instead that $c \neq 0$ and that B was obtained from A by multiplying row i with c. Prove $\mathbf{R}(A) = \mathbf{R}(B)$.

d) Assume that we bring A into reduced row echolon form R via elimination. Use the previous subtasks to argue that $\mathbf{R}(A) = \mathbf{R}(R)$.

4. Subspaces (★★☆)

Let V be a vector space and let U and W be subspaces of V. Show that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

5. Symmetric matrices (★★☆)

Let $n \in \mathbb{N}^+$ be arbitrary. Consider the set of symmetric $n \times n$ matrices $\mathcal{S}_n$ which is a subspace of $\mathbb{R}^{n \times n}$. What is the dimension of $\mathcal{S}_n$?

6. 1. Which of the following statements is true for all $n \times n$ matrices A?

(a) $\mathbf{N}(A) = \mathbf{N}(2A)$

(b) $\mathbf{N}(A) = \mathbf{N}(A^2)$

(c) $\mathbf{N}(A) = \mathbf{N}(A + I)$

(d) $\mathbf{N}(A) = \mathbf{N}(A^\top)$

2. Which of the following statements is true for all square matrices A?

(a) $\mathbf{C}(A) = \mathbf{C}(2A)$

(b) $\mathbf{C}(A) = \mathbf{C}(A^2)$

(c) $\mathbf{C}(A) = \mathbf{C}(A + I)$

(d) $\mathbf{C}(A) = \mathbf{C}(A^\top)$

3. The following equations each describe a plane in $\mathbb{R}^3$:

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 0 \\ 2x & - & 5y & + & 3z & = & 0 \\ 3x & & & + & 4z & = & 0. \end{array}$$

Which of the following statements is true?

(a) The intersection of all three planes is empty.

(b) The intersection of all three planes contains exactly one element.

(c) The intersection of all three planes is a line.

4. Consider the linear system

$$x_1 + (b-1)x_2 = 3$$
$$-3x_1 - (2b-8)x_2 = -5$$

with variables x_1, x_2 and parameter $b \in \mathbb{R}$. For which values of b is the set of solutions to the above system empty (i.e. there is no solution)?

(a) Only for $b = 0$.

(b) Only for $b = -5$.

(c) For all possible values of b (i.e. for all of $\mathbb{R}$).

(d) The system always has a solution regardless of the value of b.