D-INFK Linear Algebra HS 2023
Afonso Bandeira
Bernd Gärtner

Assignment 7

Submission Deadline: **14 November, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. **Orthogonal subspaces (hand-in)** (★☆☆)

   The exercises of this task can also be found in Section 4.1 of the blackboard notes.

   Let V, W be orthogonal subspaces of $\mathbb{R}^n$.

   a) Prove that $V \cap W = \{\mathbf{0}\}$.

   b) Recall that $V^\perp$ is the set of all vectors orthogonal to V, i.e.

   $$V^\perp := \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}.$$

   Prove that $V^\perp$ is a subspace of $\mathbb{R}^n$.

2. **The matrix product $A^\top A$** (★★☆)

   This task includes Challenge 9 from the lecture notes of the second part of the course.

   Let $A \in \mathbb{R}^{m\times n}$ be arbitrary.

   a) Prove that $\text{rank}(A) = \text{rank}(A^\top A)$.

   b) Use a) to prove $\text{rank}(A^\top) = \text{rank}(A^\top A)$ and $\text{rank}(A^\top) = \text{rank}(AA^\top)$.

   c) Prove Proposition 4.3.2 from the lecture notes, i.e. prove that $\mathbf{C}(A^\top) = \mathbf{C}(A^\top A)$.

3. **Projections** (★★★)

   This task includes Challenge 5 and Challenge 6 from the lecture notes of the second part of the course.

   Let $A \in \mathbb{R}^{m\times n}$ be an arbitrary matrix with full column rank n. In particular, this implies $n \le m$. Let $P \in \mathbb{R}^{m\times m}$ be the projection matrix associated with A, i.e. $P = A(A^\top A)^{-1}A^\top$.

   a) Compute $(I - P)^2$ and simplify the expression you get as much as possible.

   b) Let $\mathbf{w} \in \mathbf{C}(A) \subseteq \mathbb{R}^m$ be arbitrary. Recall from the lecture that P projects vectors in $\mathbb{R}^m$ to the subspace $\mathbf{C}(A)$. What do we get for $P\mathbf{w}$ and $(I - P)\mathbf{w}$?

   c) Let $\mathbf{v} \in \mathbb{R}^m$ be a vector that is orthogonal to all columns of A, i.e. $\mathbf{v} \in \mathbf{C}(A)^\perp$. Compute $(I - P)\mathbf{v}$ and simplify the expression you get as much as possible.

d) How does the rank of P compare to $\dim(\mathbf{C}(A))$?

e) How does the rank of $(I - P)$ compare to $\dim(\mathbf{C}(A)^\perp)$?

f) Prove that $(I - P)$ is the projection matrix that projects vectors from $\mathbb{R}^m$ to the subspace $\mathbf{C}(A)^\perp$.

4. Application of normal equations (★☆☆) This exercise might be slightly ahead of the lecture. But you should definitely be able to solve it after the lecture on Friday, 10th of November.

In this task, we want to determine the parameters of a certain model function from a few measured values. In particular, assume that we measured the following values

x_i	1	2	3	4	5
y_i	2	3	5	6	8

where $i \in [5]$. Moreover, assume that we want to model the relationship between x, y by a function f, i.e. $y = f(x)$. We have seen before (in previous assignments) that we could choose f to be a polynomial of large enough degree to then interpolate all datapoints. But depending on the application, choosing f to be a high degree polynomial might not be desirable. In particular, we might want to restrict the degree of f. In this exercise, we restrict f to be a line, i.e. f should have the form

$$f(x) = ax + b$$

for parameters $a, b \in \mathbb{R}$. Our goal is to find suitable values for a, b.

a) For each datapoint (x_i, y_i) with $i \in [5]$, we get an equation for a, b from $f(x_i) = y_i$. Write down the system of linear equations that we get by combining all five equations.

b) Do you expect this system to have any solutions? (Answer this intuitively without actually solving the system).

c) Using the normal equations, find an approximate solution to the system you wrote down.