D-INFK Linear Algebra HS 2023
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Assignment 9

Submission Deadline: **28 November, 2023** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA23

Exercises

You can get feedback from your TA for Exercise 1 by handing in your solution as pdf via Moodle before the deadline.

1. **Properties of pseudoinverses (hand-in)** (★★☆)

   This is Challenge 23 from the lecture notes.

   Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ be arbitrary matrices.

   a) Prove that if $\text{rank}(A) = \text{rank}(B) = n$, we have $(AB)^\dagger = B^\dagger A^\dagger$.

   b) Prove that $(A^\top)^\dagger = (A^\dagger)^\top$.

   c) Prove that $AA^\dagger$ is symmetric and that it is the projection matrix for the subspace $\mathbf{C}(A)$.

   d) Prove that $A^\dagger A$ is symmetric and that it is the projection matrix for the subspace $\mathbf{C}(A^\top)$.

   Hint: Use Proposition 4.5.9 from the lecture notes.

2. **Bijective map** (★★☆)

   This task includes Challenge 24 from the lecture notes which asks you to prove Proposition 4.5.11.

   Let $A \in \mathbb{R}^{m\times n}$ be an arbitrary matrix with column space $\mathbf{C}(A)$ and row space $\mathbf{C}(A^\top)$. Consider the function $f : \mathbf{C}(A^\top) \to \mathbf{C}(A)$ that maps $\mathbf{x} \in \mathbf{C}(A^\top)$ to $(A\mathbf{x}) \in \mathbf{C}(A)$. Prove that f is bijective.

   Hint: Solve Exercise 1 first.

3. **Linear transformations** (★☆☆)

   This task provides some ideas for Challenge 26, but we encourage you to find some more examples on your own. For all subtask it might help to draw a sketch of the respective linear transformations to understand what is going on.

   a) Let $\mathbf{v} \in \mathbb{R}^2$ be a unit vector. Consider the linear transformation given by the matrix $A = I - 2\mathbf{v}\mathbf{v}^\top$. Geometrically speaking, does applying A correspond to stretching, shearing, rotating, or reflecting vectors?

   b) Consider the linear transformation given by the matrix

   $$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

   Describe the geometric operation that this linear transformation corresponds to.

c) Consider the line $L = \{c\mathbf{v} : c \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with $\mathbf{v} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$. Find a matrix $A \in \mathbb{R}^{3\times 3}$ that corresponds to rotating vectors by $180°$ around the axis L.

Hint: Try to find out what $A\mathbf{e}_1$, $A\mathbf{e}_2$, and $A\mathbf{e}_3$ should be.

4. Linear transformation of triangles (★★☆)

This task includes Challenge 27 from the lecture notes.

For this exercise, we will need the notion of a line segment. Consider an arbitrary set $S \subseteq \mathbb{R}^2$. We say that S is a line segment if and only if there exist two distinct points $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ such that

$$S = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 : c_1, c_2 \in \mathbb{R}_0^+, c_1 + c_2 = 1\}.$$

In other words, S is the set of so-called *convex combinations* of $\mathbf{v}_1$ and $\mathbf{v}_2$. Notice that in contrast to linear combinations, for convex combinations we additionally require the coefficients to be non-negative and their sum to be 1. Try to convince yourself that this characterization of line segments corresponds to what you intuitively think of as a line segment. It might help to draw some examples.

We will also need a more concrete notion of a triangle: We say that a set $T \subseteq \mathbb{R}^2$ is a triangle if and only if T is not a line segment and there exist three distinct points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$ such that

$$T = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 : c_1, c_2, c_3 \in \mathbb{R}_0^+, c_1 + c_2 + c_3 = 1\}.$$

Again, convince yourself that this characterization intuitively makes sense by drawing some examples.

Let now T be an arbitrary triangle in the plane with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^2$. Let $A \in \mathbb{R}^{2\times 2}$ be an arbitrary matrix and consider the set

$$T' = \{c_1\mathbf{v}_1' + c_2\mathbf{v}_2' + c_3\mathbf{v}_3' : c_1, c_2, c_3 \in \mathbb{R}_0^+, c_1 + c_2 + c_3 = 1\}$$

with $\mathbf{v}_1' = A\mathbf{v}_1$, $\mathbf{v}_2' = A\mathbf{v}_2$, and $\mathbf{v}_3' = A\mathbf{v}_3$.

a) Prove that $(A\mathbf{x}) \in T'$ for all $\mathbf{x} \in T$.

b) Prove that if T' is neither a single point nor a triangle, then it has to be a line segment.

c) Prove that A has rank 0 if and only if T' is a point.

d) Prove that A has rank 2 if and only if T' is a triangle.

e) Prove that A has rank 1 if and only if T' is a line segment.

Hint: Use previous subtasks.

5. Fitting a circle (★☆☆)

Consider the following points

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \in \mathbb{R}^2$$

in the plane. We want to find a circle C_r with origin $\mathbf{0}$ and radius $r \in \mathbb{R}^+$ such that the sum of the quadratic distances of the points to the circle is minimized. Note that the quadratic distance of a point $\mathbf{p} \in \mathbb{R}^2$ to the circle C_r is $(r - ||\mathbf{p}||)^2$. Find the optimal value of r for the four points above.

Note that the interesting thing here is to find a formula for r in terms of $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. The actual numerical answer is of secondary interest here, i.e. you are not expected to simplify the value you get for r as much as possible.