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# Solution for Assignment 0

## 1. Linear combinations of vectors (★☆☆)

**a)** Consider an arbitrary vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$. We define $c := \frac{b_1+b_2}{2}$ and $d := \frac{b_2-b_1}{2}$. Then we have

$$c\mathbf{v} + d\mathbf{w} = \begin{bmatrix} c-d \\ c+d \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mathbf{b} \quad (1)$$

which proves that $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$. While this proves the claim and is considered a complete solution, let us also explain how to find out that we should pick $c := \frac{b_1+b_2}{2}$ and $d := \frac{b_2-b_1}{2}$. For this, assume that we do not know yet what values $c$ and $d$ should have. We can still write down Equation 1, interpreting $c$ and $d$ as unknowns. In fact, this gives us a system of two equations

$$\begin{aligned} c - d &= b_1 \\ c + d &= b_2 \end{aligned}$$

(one for each coordinate). Adding the two equations we obtain $2c = b_1 + b_2$ and hence $c = \frac{b_1+b_2}{2}$. On the other hand, subtracting the first equation from the second gives $2d = b_2 - b_1$ and hence $d = \frac{b_2-b_1}{2}$. One can also think about this geometrically, analogous to the "row picture" from Section 1.1.3 in the blackboard notes (Figure 1).

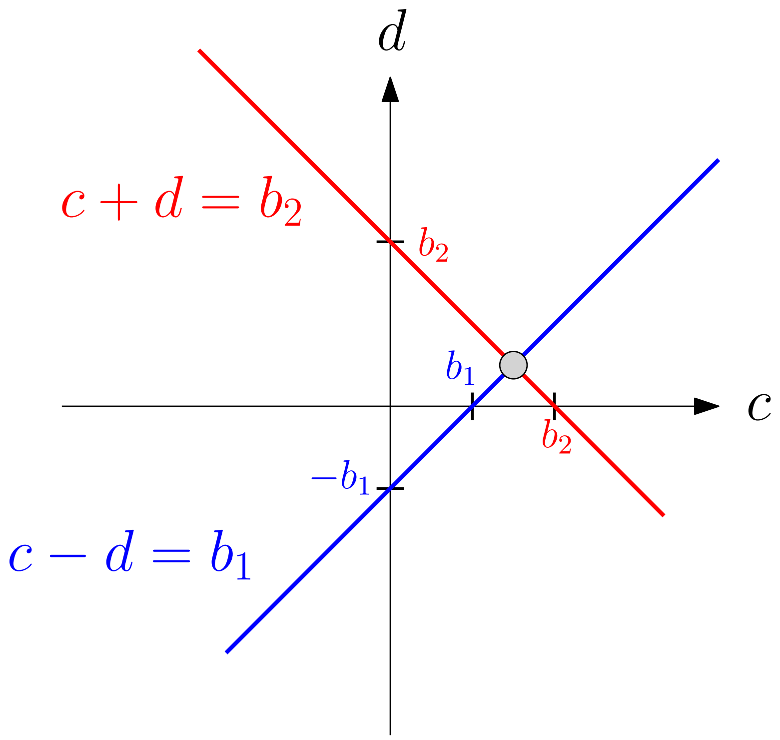


Figure 1: The lines given by the two equations intersect at $(\frac{b_1+b_2}{2}, \frac{b_2-b_1}{2})$.

**b)** Consider the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{R}^3$. In order to write it as a linear combination of $\mathbf{v}$ and $\mathbf{w}$, we would have to find $c, d \in \mathbb{R}$ such that

$$c\mathbf{v} + d\mathbf{w} = c \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{b}.$$

Each coordinate provides a constraint on the values of $c$ and $d$. In particular, from the first coordinate we get the constraint $c = 0$, from the second coordinate we get the constraint $d = 1$, and from the third coordinate we get $c + d = 0$. This system of three equations in two unknowns does not have a solution and hence we conclude that $\mathbf{b}$ cannot be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

**2. The perfect long drink** (★☆☆)

**a)** Mixing $\frac{3}{5}$ of the first imperfect drink with $\frac{2}{5}$ of the second imperfect drink will yield a perfect drink. To see this, represent the imperfect drinks as vectors $\begin{bmatrix} 15 \\ 85 \end{bmatrix} \in \mathbb{R}^2$ and $\begin{bmatrix} 35 \\ 65 \end{bmatrix} \in \mathbb{R}^2$, and the perfect drink as $\begin{bmatrix} 23 \\ 77 \end{bmatrix} \in \mathbb{R}^2$. It remains to check that

$$\frac{3}{5} \begin{bmatrix} 15 \\ 85 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 35 \\ 65 \end{bmatrix} = \begin{bmatrix} 23 \\ 77 \end{bmatrix}.$$

While this is already a full solution, we describe again how one could arrive at $\frac{3}{5}$ and $\frac{2}{5}$, respectively. We want to find $c$ and $d$ such that

$$c \begin{bmatrix} 15 \\ 85 \end{bmatrix} + d \begin{bmatrix} 35 \\ 65 \end{bmatrix} = \begin{bmatrix} 23 \\ 77 \end{bmatrix}.$$

This gives us the two equations

$$15c + 35d = 23$$
$$85c + 65d = 77$$

which can be solved in various ways. One way is to add the two equations to get $100c+100d = 100$ and thus $c + d = 1$. Now plug in $c = 1 - d$ into the first equation to get $d = \frac{2}{5}$ and hence $c = 1 - d = \frac{3}{5}$.

**b)** The set $\hat{D}$ can be written down as

$$\hat{D} = \{c\mathbf{v} + d\mathbf{w} \in \mathbb{R}^2 : c \geq 0, d \geq 0, c + d = 1\}.$$

In words, $\hat{D}$ consists of all vectors that can be written as linear combinations $c\mathbf{v} + d\mathbf{w}$ that additionally satisfy $c \geq 0$, $d \geq 0$ and $c + d = 1$. We need the constraints $c \geq 0$ and $d \geq 0$ because we cannot use a negative amount of one of the drinks. Moreover, we need to make sure that we are mixing a drink of size $100\text{ml}$, hence the constraint $c + d = 1$. In math lingo, $\hat{D}$ is the set of convex combinations of $\mathbf{v}$ and $\mathbf{w}$. Geometrically, $\hat{D}$ has the shape of a line segment that connects the points $\mathbf{v}$ and $\mathbf{w}$.

**c)** Since the mixed drink does not need to contain exactly $100\text{ml}$ anymore, we can get rid of the constraint $c + d = 1$. But we still cannot use $\mathbf{v}$ and $\mathbf{w}$ more than once each, i.e. we need to include the constraints $c \leq 1$ and $d \leq 1$ instead. We get the parallelogram (see Figure 2)

$$\overline{D} = \{c\mathbf{v} + d\mathbf{w} \in \mathbb{R}^2 : 0 \leq c \leq 1, 0 \leq d \leq 1\}.$$

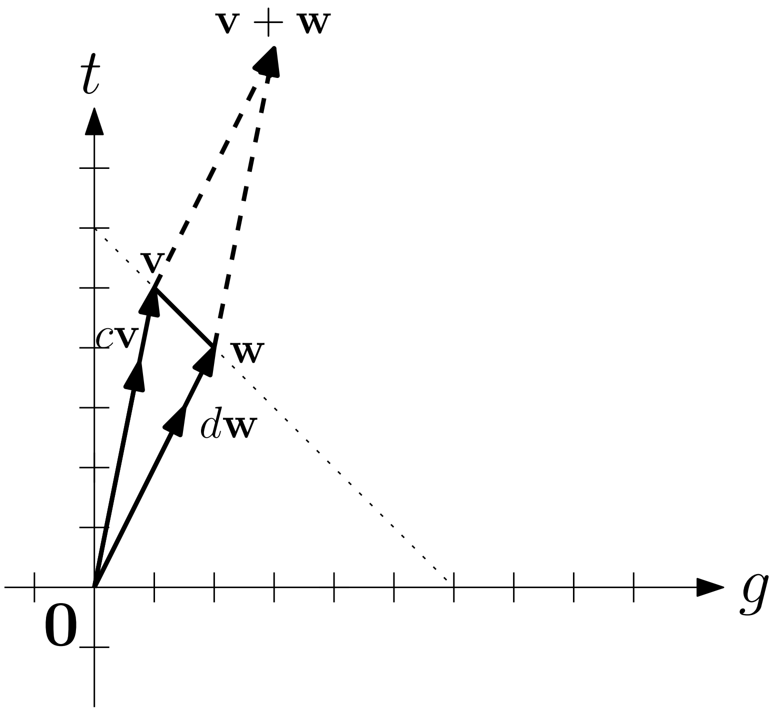


Figure 2: A sketch of the situation with the two imperfect drinks $\mathbf{v}$ and $\mathbf{w}$. The set $D$ is shown as a dotted line between the two axis. The set $\hat{D}$ is shown in bold connecting $\mathbf{v}$ and $\mathbf{w}$. The set $\overline{D}$ is the parallelogram. We can see in the picture that any linear combination $c\mathbf{v} + d\mathbf{w}$ with $0 \leq c \leq 1$ and $0 \leq d \leq 1$ must land inside the parallelogram. Moreover, any point inside the parallelogram can be reached by choosing $c$ and $d$ appropriately. The argument for this is given in the "column picture" from Section 1.1.3 of the first lecture.