Solution for Assignment 1

1. **a)** This set of vectors is not linearly independent as $\mathbf{v}$ can always be written as a linear combination of any other vector, e.g. $0\mathbf{u} = \mathbf{v}$.

b) This set of vectors is linearly independent. To see this, we first check that $\mathbf{v}$ cannot be obtained as a linear combination of $\mathbf{u}$: indeed, observe that we cannot obtain the 1 in the third and fourth coordinate of $\mathbf{v}$ from $\mathbf{u}$ as $\mathbf{u}$ contains 0 in both of those coordinates.

Next, we check that $\mathbf{w}$ cannot be obtained as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Such a linear combination would require scalars c and d such that

$$c\mathbf{u} + d\mathbf{v} = c\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\1\\1\end{bmatrix} \overset{!}{=} \begin{bmatrix}0\\1\\1\\0\end{bmatrix} = \mathbf{w}.$$

But notice that the 1 in the second coordinate of $\mathbf{w}$ can only be obtained by setting $c = 1$. Similarly, the 1 in the third coordinate of $\mathbf{w}$ can only be obtained with $d = 1$. But with this choice of c and d, we would get 1 (instead of 0) in the first and forth coordinate. So we conclude that there is no such linear combination.

From the lecture we know that checking the three vectors in any order (we did it in the order they were given) suffices to check linear independence. Hence, we conclude that the three vectors are linearly independent.

c) In order to determine the rank of A, we have to find out how many of its columns are linearly independent. Let us denote the columns of A by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, i.e.

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}.$$

By checking the column vectors in order we quickly see that $\mathbf{v}_2$ can be obtained from $\mathbf{v}_1$ as $\mathbf{v}_2 = -3\mathbf{v}_1$. But if we try to obtain $\mathbf{v}_3$ from $\mathbf{v}_1$ we fail, since there is no $c \in \mathbb{R}$ such that both $1c = 3$ and $-2c = 0$. In other words, the set of vectors $\{\mathbf{v}_1, \mathbf{v}_3\}$ is linearly independent. We found a set of two vectors that is linearly independent and also observed that the set of all three vectors is not linearly independent. Therefore, the rank of A is 2.

d) We proceed as in subtask c) and check linear independence of the columns of A. Let us again denote the columns of A by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, i.e.

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}.$$

We choose to check the columns in reverse order $\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1$. In particular, we first check whether $\mathbf{v}_2$ is independent from $\mathbf{v}_3$. Indeed, there is no way of obtaining the 1 in the first coordinate of $\mathbf{v}_2$ if we only use $\mathbf{v}_3$, hence $\mathbf{v}_2$ is linearly independent from $\mathbf{v}_3$. It remains to check whether $\mathbf{v}_1$ can be obtained as linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. Here we observe that there is no way of obtaining the 2 in the second coordinate of $\mathbf{v}_1$ since the second entry of both $\mathbf{v}_2$ and $\mathbf{v}_3$ is 0. We conclude that the three vectors are linearly independent and hence A has rank 3.

2. **a)** By definition of matrix-vector multiplication, we have

$$A\mathbf{x} = \begin{bmatrix} — & \mathbf{u}_1 & — \\ — & \mathbf{u}_2 & — \\ & \vdots & \\ — & \mathbf{u}_m & — \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{x} \\ \mathbf{u}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{u}_m \cdot \mathbf{x} \end{bmatrix}.$$

Assume now that we have $A\mathbf{x} = \mathbf{0}$. Then we must have $\mathbf{u}_i \cdot \mathbf{x} = 0$ for all $1 \le i \le m$. By definition, this means that $\mathbf{x}$ is perpendicular to each of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$. Conversely, if $\mathbf{x}$ is perpendicular to each of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, we must have $\mathbf{u}_i \cdot \mathbf{x} = 0$ for all $1 \le i \le m$ and hence $A\mathbf{x} = \mathbf{0}$.

b) We take a small detour and first consider an arbitrary vector $\mathbf{v}$ that is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. By definition, this means that we have $\mathbf{v} \cdot \mathbf{x} = 0$ and $\mathbf{v} \cdot \mathbf{y} = 0$. Now consider the scalar product $\mathbf{v} \cdot (c\mathbf{x} + d\mathbf{y})$. We have

$$\begin{aligned} \mathbf{v} \cdot (c\mathbf{x} + d\mathbf{y}) &= v_1(cx_1 + dy_1) + v_2(cx_2 + dy_2) + \cdots + v_n(cx_n + dy_n) \\ &= cv_1x_1 + cv_2x_2 + \cdots + cv_nx_n + dv_1y_1 + dv_2y_2 + \cdots + dv_ny_n \\ &= c\,\mathbf{v} \cdot \mathbf{x} + d\,\mathbf{v} \cdot \mathbf{y} \\ &= 0. \end{aligned}$$

In other words, any vector $\mathbf{v}$ that is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$ is also perpendicular to $(c\mathbf{x}+d\mathbf{y})$. Now consider an arbitrary row $\mathbf{u}_i$ from A. By subtask a), we know that $\mathbf{u}_i$ is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$ since we have $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. Hence, $\mathbf{u}_i$ must also be perpendicular to $(c\mathbf{x} + d\mathbf{y})$.

c) No, $\mathcal{L}$ is not a finite set. Take any non-zero vector $\mathbf{x} \in \mathcal{L}$ (which must exist since $|\mathcal{L}| \ge 2$). Then every row $\mathbf{u}_i$ of A must be perpendicular to $\mathbf{x}$ by subtask a). Hence, $\mathbf{u}_i$ must also be perpendicular to $c\mathbf{x}$ for any choice of $c \in \mathbb{R}$ by subtask b). In other words, $c\mathbf{x}$ is perpendicular to all rows of A. By using the result from subtask a) again, this implies that $A(c\mathbf{x}) = 0$ and hence $c\mathbf{x} \in \mathcal{L}$. For every choice of c we obtain a different vector in $\mathcal{L}$. The number of choices of c is infinite/unbounded, hence $\mathcal{L}$ is not a finite set.

3. We have $A = E$, $B = F$ and $C = D$ and also $A \ne B \ne C \ne A$. The equality of matrices B and F is observed quickly since $b_{ij} = i + j = (i+1) + (j-1) = f_{ij}$. The equality of the matrices A and E can be seen by using the binomial formula:

$$e_{ij} = i\left(\frac{j^2-1}{j+1} + 1\right) = i\left(\frac{(j-1)(j+1)}{j+1} + 1\right) = ij = a_{ij}.$$

For the last equality, we distinguish the two cases $j \ge i$ and $j < i$. In the first case, we have $|i-j| = j-i$ and hence

$$d_{ij} = \frac{i+j}{2} + \frac{|i-j|}{2} = j = \max\{i,j\} = c_{ij}.$$

In the second case, we instead have $|i-j| = i-j$ and

$$d_{ij} = \frac{i+j}{2} + \frac{|i-j|}{2} = i = \max\{i,j\} = c_{ij}.$$

It remains to prove $A \ne B \ne C \ne A$. For this, we simply observe that $a_{3,3} = 9$, $b_{3,3} = 6$, and $c_{3,3} = 3$.

4. **a)** By definition of L, there exists $\mathbf{w} \in \mathbb{R}^n$ such that $L = \{c\mathbf{w} : c \in \mathbb{R}\}$. In particular, we can write $\mathbf{u} = c_u\mathbf{w}$ for some $c_u \in \mathbb{R}$. Observe that we have $c_u \ne 0$ because $\mathbf{u} \ne \mathbf{0}$. Now consider an arbitrary vector $\mathbf{v} \in L$. There must be $c_v \in \mathbb{R}$ such that $\mathbf{v} = c_v\mathbf{w}$. Putting these together, we get $\mathbf{v} = \frac{c_v}{c_u}\mathbf{u}$. This already proves $L \subseteq \{c\mathbf{u} : c \in \mathbb{R}\}$ and it remains to prove $\{c\mathbf{u} : c \in \mathbb{R}\} \subseteq L$. For this, consider an arbitrary vector $\mathbf{v}' = c_{v'}\mathbf{u} \in \{c\mathbf{u} : c \in \mathbb{R}\}$. Combining $\mathbf{v}' = c_{v'}\mathbf{u}$ with $\mathbf{u} = c_u\mathbf{w}$ we get $\mathbf{v}' = c_{v'}c_u\mathbf{w}$ and hence $\mathbf{v}' \in L$. We conclude that $L = \{c\mathbf{u} : c \in \mathbb{R}\}$.

b) Let L_1 and L_2 be two lines of $\mathbb{R}^n$. By definition, there exist $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $L_1 = \{c\mathbf{w}_1 : c \in \mathbb{R}\}$ and $L_2 = \{c\mathbf{w}_2 : c \in \mathbb{R}\}$. In order to see that $\mathbf{0} \in L_1 \cap L_2$, it suffices to observe $\mathbf{0} \in L_1$ and $\mathbf{0} \in L_2$ since we have $\mathbf{0} = 0\mathbf{w}_1 = 0\mathbf{w}_2$. Now assume $L_1 \cap L_2 \neq \{\mathbf{0}\}$. Because $L_1 \cap L_2$ is not empty, there exists a non-zero vector $\mathbf{u} \in L_1 \cap L_2$. By $\mathbf{u} \in L_1$, we know from part a) that $L_1 = \{c\mathbf{u} : c \in \mathbb{R}\}$. Analogously, we have $L_2 = \{c\mathbf{u} : c \in \mathbb{R}\}$ and hence $L_1 = L_2$.

c) By definition, there must be a vector $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$ such that $L = \{c\mathbf{w} : c \in \mathbb{R}^2\}$. We want to find a vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathbb{R}^2$ such that $L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$. In particular, we want $\mathbf{w} \cdot \mathbf{d} = w_1 d_1 + w_2 d_2 \overset{!}{=} 0$ since $\mathbf{w} \in L$. Choosing $d_1 := -w_2$ and $d_2 := w_1$ would certainly work, so let this be our "guess" for $\mathbf{d}$. It remains to prove that with this choice of $\mathbf{d}$, we have $L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$.

$\subseteq$: Consider an arbitrary element $\mathbf{u} = c_u\mathbf{w} \in L$. We have

$$\mathbf{u} \cdot \mathbf{d} = (c_u\mathbf{w}) \cdot \mathbf{d} = c_u w_1 d_1 + c_u w_2 d_2 = c_u(w_1 d_1 + w_2 d_2) = 0$$

and hence $\mathbf{u} \in \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$.

$\supseteq$: Consider an arbitrary element $\mathbf{v} \in \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$. In particular, we have $v_1 d_1 + v_2 d_2 = -v_1 w_2 + v_2 w_1 = 0$. Our goal is to find c such that $\mathbf{v} = c\mathbf{w}$. Recall from the definition of a line that we must have $\mathbf{w} \neq \mathbf{0}$ and hence either $w_1 \neq 0$ or $w_2 \neq 0$. Assume first $w_1 \neq 0$ and observe that we can rewrite $-v_1 w_2 + v_2 w_1 = 0$ to $v_2 = \frac{w_2}{w_1} v_1$. Choosing $c = \frac{v_1}{w_1}$ we can see that indeed, we have $v_1 = cw_1$ and $v_2 = cw_2$, as desired. If we have $w_1 = 0$, then it must be the case that $w_2 \neq 0$. But then we can rewrite $-v_1 w_2 + v_2 w_1 = 0$ to $v_1 = \frac{w_1}{w_2} v_2$ and choose $c = \frac{v_2}{w_2}$.

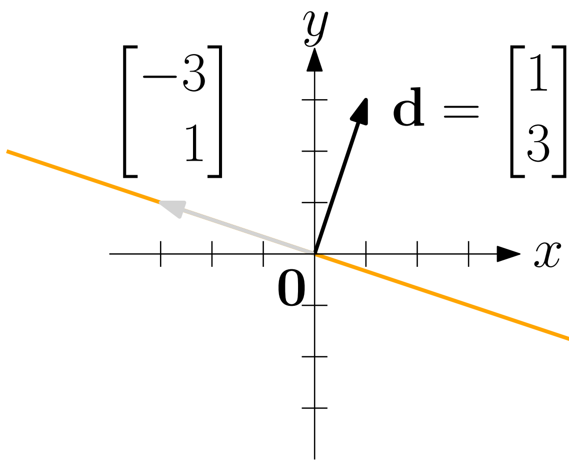

Figure 1: This figure from the lecture notes illustrates the situation in task c). The yellow line given by $\{c \begin{bmatrix} -3 \\ 1 \end{bmatrix} : c \in \mathbb{R}\}$ is equal to the hyperplane $\{\mathbf{v} : \mathbf{v} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0\}$.

5. We first observe that we have $\|\mathbf{v}\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{z^2 + x^2 + y^2} = \|\mathbf{w}\|$. In particular, this implies $\|\mathbf{v}\| \|\mathbf{w}\| = x^2 + y^2 + z^2$. Using the formula from the lecture for the angle α between $\mathbf{v}$ and

$\mathbf{w}$, we calculate

$$\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\, \mathbf{v} \,\| \, \|\, \mathbf{w} \,\|} = \frac{xz + yx + zy}{x^2 + y^2 + z^2}.$$

Next, observe that we can rewrite $xz + yx + zy = \frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. By our assumption $x + y + z = 0$, the first term vanishes and we obtain

$$\cos(\alpha) = \frac{xz + yx + zy}{x^2 + y^2 + z^2} = \frac{-\frac{1}{2}(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2} = -\frac{1}{2}.$$

To find α, it remains to look up (or remember) $\cos^{-1}(-\frac{1}{2}) = \frac{2}{3}\pi$ ($= 120°$).