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Solution for Assignment 10

1. **a)** Let C_{ij} be the co-factors of A where $i, j \in [5]$. Note that by combining Propositions 5.1.13 and 5.1.9, we get

$$\begin{aligned}
\mathsf{det}A &\overset{5.1.9}{=} \mathsf{det}A^\top \\
&\overset{5.1.13}{=} \sum_{j=1}^{5} (A^\top)_{3,j}(C^\top)_{3,j} \\
&= \sum_{i=1}^{5} A_{i,3}C_{i,3} \\
&= 0C_{1,3} + 0C_{2,3} + bC_{3,3} + 0C_{4,3} + 0C_{5,3} \\
&= b \cdot (-1)^{(3+3)} \cdot \begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix}.
\end{aligned}$$

This is also sometimes called *expansion of the determinant along the third column*. In particular, we chose the third column because it contains many zeroes and hence many terms disappeared. In order to compute

$$\begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix}$$

we use the same trick again for the first column. In this way we obtain

$$\begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix} = a \cdot (-1)^{(2+1)} \begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix}.$$

We repeat this one more time for the third column of

$$\begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix}$$

to get

$$\begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix} = c \cdot (-1)^{(1+3)} \cdot \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} + 1 \cdot (-1)^{(3+3)} \cdot \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix}.$$

We can compute these 2×2 determinants using the formula from Proposition 5.1.3 to get

$$\begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -2 \text{ and } \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} = 9.$$

Putting everything together, we obtain

$$\mathsf{det}A = b \cdot (-1)^{(3+3)} \left(a \cdot (-1)^{(2+1)} \left(c \cdot (-1)^{(1+3)} \cdot (-2) + 1 \cdot (-1)^{(3+3)} \cdot 9 \right) \right) = ab(2c - 9).$$

We conclude that $\mathsf{det}A = 0$ if and only if $a = 0$, or $b = 0$, or $c = \frac{9}{2}$.

b) As it turns out, we only need to perform one step of Gauss elimination on B to obtain U:

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & 0 \\ -1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 6 \\ 0 & 0 & -1 \end{bmatrix} =: U.$$

Using Proposition 5.1.8, we see that $\mathsf{det}(U) = -2$. By Proposition 5.1.19 (or the discussion in Section 5.1.6), we know that the determinant of U is the same as the determinant of B (we did not swap any rows). Hence, we conclude $\mathsf{det}(B) = -2$.

2. **a)** As the hint suggests, we start by using Definition 5.1.6 for the determinant of M, i.e. we have

$$\mathsf{det} M = \sum_{\sigma \in \Pi_n} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$$

where Π_n is the set of all permutations on n elements. The key observation for this exercise is that only those permutations $\sigma \in \Pi_n$ that satisfy $\sigma(1), \ldots, \sigma(m) \in \{1, \ldots, m\}$ will contribute to this sum. To see this, let $\sigma \in \Pi_n$ be a permutation with $\sigma(i) > m$ for some $i \in [m]$. By the pigeonhole principle, there must exist $j \in [n] \setminus [m]$ with $\sigma(j) \in [m]$. But by the shape of M, we must have $M_{j,\sigma(i)} = 0$ and hence the contribution of σ to the sum is 0.

In particular, the relevant (those that contribute non-zero terms to the sum) permutations $\sigma \in \Pi_n$ satisfy $\sigma(i) \in [m]$ for all $i \in [m]$ and $\sigma(j) \in [n] \setminus [m]$ for all $j \in [n] \setminus [m]$. In other words, restricting such a permutation σ to $[m]$ yields a permutation on m elements, and restricting σ to $[n] \setminus [m]$ yields a permutation on $n - m$ elements. Conversely, any two permutations $\sigma_1 \in \Pi_m$ and $\sigma_2 \in \Pi_{n-m}$ yield a permutation $\sigma \in \Pi_n$ that contributes to the sum (define $\sigma(i) = \sigma_1(i)$ for $i \in [m]$ and $\sigma(j) = m + \sigma_2(j - m)$ for $j \in [n] \setminus [m]$). Observe that the number of inversions in σ is exactly the number of inversions in σ_1 plus the number of inversions in σ_2. Hence, we always have $\mathsf{sign}(\sigma) = \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2)$ in this correspondence.

We conclude that we can rewrite the sum as

$$\mathsf{det} M = \sum_{\sigma \in \Pi_n} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)} = \sum_{\sigma_1 \in \Pi_m} \sum_{\sigma_2 \in \Pi_{n-m}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2) \prod_{i=1}^{m} M_{i,\sigma_1(i)} \prod_{j=m+1}^{n} M_{j,j+\sigma_2(j-m)}.$$

Next, observe that the terms $M_{i,\sigma_1(i)}$ are always in the A-part of M, i.e. we have $M_{i,\sigma_1(i)} = A_{i,\sigma_1(i)}$. Similarly, the terms $M_{j,j+\sigma_2(j-m)}$ are always in the C-part of M, i.e. we have $M_{j,j+\sigma_2(j-m)} = C_{j-m,\sigma_2(j-m)}$. Hence, we can further rewrite the sum as

$$\begin{aligned}
\mathsf{det} M &= \sum_{\sigma_1 \in \Pi_m} \sum_{\sigma_2 \in \Pi_{n-m}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2) \prod_{i=1}^{m} M_{i,\sigma_1(i)} \prod_{j=m+1}^{n} M_{j,j+\sigma_2(j-m)} \\
&= \sum_{\sigma_1 \in \Pi_m} \sum_{\sigma_2 \in \Pi_{n-m}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \prod_{j=m+1}^{n} C_{j-m,\sigma_2(j-m)} \\
&= \sum_{\sigma_1 \in \Pi_m} \mathsf{sign}(\sigma_1) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \left(\sum_{\sigma_2 \in \Pi_{n-m}} \mathsf{sign}(\sigma_2) \prod_{j=m+1}^{n} C_{j-m,\sigma_2(j-m)} \right) \\
&= \left(\sum_{\sigma_1 \in \Pi_m} \mathsf{sign}(\sigma_1) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \right) \left(\sum_{\sigma_2 \in \Pi_{n-m}} \mathsf{sign}(\sigma_2) \prod_{j=m+1}^{n} C_{j-m,\sigma_2(j-m)} \right) \\
&= \left(\sum_{\sigma_1 \in \Pi_m} \mathsf{sign}(\sigma_1) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \right) \left(\sum_{\sigma_2 \in \Pi_{n-m}} \mathsf{sign}(\sigma_2) \prod_{j=1}^{n-m} C_{j,\sigma_2(j)} \right) \\
&= \mathsf{det}(A)\mathsf{det}(C)
\end{aligned}$$

which concludes the proof.

b) In order to calculate the determinant of M using the previous result, we must first bring it into the right form. Clearly, M already contains a lot of zero entries. In the end, we want to have a block of zeroes in the bottom left corner. We can use that transposing the matrix does not change its determinant. Moreover, by Proposition 5.1.18, swapping two rows of a matrix negates its determinant. Hence we proceed as follows: we first transpose M and then swap the second row and fourth row, as well as the third and sixth row of the resulting matrix. In this way, we obtain the matrix

$$M' = \begin{bmatrix} 2 & 9 & 1 & 3 & 2 & 8 \\ 4 & 0 & 0 & 5 & 5 & 3 \\ 7 & 4 & 0 & 7 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Using the result from the previous subtask and some more row swaps as well as the formula for the determinant of triangular matrices, we get

$$\begin{aligned} \mathsf{det} M &= (-1)^2 \mathsf{det} M' \\ &= \mathsf{det} M' \\ &= \begin{vmatrix} 2 & 9 & 1 \\ 4 & 0 & 0 \\ 7 & 4 & 0 \end{vmatrix} \begin{vmatrix} 2 & 3 & 8 \\ 0 & 0 & 2 \\ 0 & 1 & 7 \end{vmatrix} \\ &= (-1)^2 \begin{vmatrix} 4 & 0 & 0 \\ 7 & 4 & 0 \\ 2 & 9 & 1 \end{vmatrix} (-1) \begin{vmatrix} 2 & 3 & 8 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{vmatrix} \\ &= -16 \cdot 4 = -64. \end{aligned}$$

3. **a)** We prove this by induction over k.

- **Property**: $T(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k) = \alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_k T(\mathbf{u}_k)$ for all $\mathbf{u}_1, \ldots, \mathbf{u}_k \in U$ and all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$.
- **Base case**: For $k = 1$, the property is true by linearity of T.
- **Induction step**: Fix a natural number $1 \leq k$ and assume that the property is true for k (induction hypothesis). We prove that the property is true for $k+1$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_{k+1} \in U$ and $\alpha_1, \ldots, \alpha_{k+1} \in \mathbb{R}$ be arbitrary. By linearity of T, we have

$$T(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_{k+1} \mathbf{u}_{k+1}) = T(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k) + \alpha_{k+1} T(\mathbf{u}_{k+1}).$$

  Moreover, we have

$$T(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_k \mathbf{u}_k) = \alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_k T(\mathbf{u}_k)$$

  by the induction hypothesis. Plugging both together yields the desired result

$$T(\alpha_1 \mathbf{u}_1 + \cdots + \alpha_{k+1} \mathbf{u}_{k+1}) = \alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_k T(\mathbf{u}_k) + \alpha_{k+1} T(\mathbf{u}_{k+1}).$$

b) We explicitly construct such a function T. First, we define $T(\mathbf{u}_i) = \mathbf{v}_i$ for all $i \in [n]$. Now let $\mathbf{x} \in U$ be arbitrary. Since $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis of U, there exist unique scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ with $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$. Hence, we can define $T(\mathbf{x}) = \alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_n T(\mathbf{u}_n)$. Since this works for any $\mathbf{x} \in U$ and is consistent with $T(\mathbf{u}_i) = \mathbf{v}_i$ for all $i \in [n]$, it follows that T is a well-defined function on U. It remains to argue that T is linear. For this, let $\mathbf{x}, \mathbf{y} \in U$ and

$c \in \mathbb{R}$ be arbitrary. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $\beta_1, \dots, \beta_n \in \mathbb{R}$ be the unique coefficients such that $\mathbf{x} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$ and $\mathbf{y} = \beta_1 \mathbf{u}_1 + \cdots + \beta_n \mathbf{u}_n$. We have

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= T((\alpha_1+\beta_1)\mathbf{u}_1 + \cdots + (\alpha_n+\beta_n)\mathbf{u}_n) \\ &= (\alpha_1+\beta_1)T(\mathbf{u}_1) + \cdots + (\alpha_n+\beta_n)T(\mathbf{u}_n) \\ &= (\alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_n T(\mathbf{u}_n)) + (\beta_1 T(\mathbf{u}_1) + \cdots + \beta_n T(\mathbf{u}_n)) \\ &= T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

and also

$$\begin{aligned} T(c\mathbf{x}) &= T(c\alpha_1 \mathbf{u}_1 + \cdots + c\alpha_n \mathbf{u}_n) \\ &= c\alpha_1 T(\mathbf{u}_1) + \cdots + c\alpha_n T(\mathbf{u}_n) \\ &= c\,(\alpha_1 T(\mathbf{u}_1) + \cdots + \alpha_n T(\mathbf{u}_n)) \\ &= cT(\mathbf{x}) \end{aligned}$$

by definition of T. We conclude that T is indeed linear.

c) Consider an arbitrary non-zero vector $\mathbf{x} \in \mathbb{R}^n$. If T were linear, we would have $T(-\mathbf{x}) = -T(\mathbf{x})$. But this is clearly not the case, as we have $0 \neq T(\mathbf{x}) = \| \mathbf{x} \| = \| -\mathbf{x} \| = T(-\mathbf{x})$. We conclude that T cannot be linear.

d) Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. We directly observe that

$$L \circ T(\mathbf{x}) = L(T(\mathbf{x})) = L(A\mathbf{x}) = B(A\mathbf{x}) = BA\mathbf{x}.$$

4. **a)** By definition, we have $C_{11} = (-1)^2 \mathsf{det}(\begin{bmatrix} d \end{bmatrix}) = d$. Similarly, we obtain

$$\begin{aligned} C_{12} &= (-1)^3 \mathsf{det} \begin{bmatrix} c \end{bmatrix} = -c \\ C_{21} &= (-1)^3 \mathsf{det} \begin{bmatrix} b \end{bmatrix} = -b \\ C_{22} &= (-1)^4 \mathsf{det} \begin{bmatrix} a \end{bmatrix} = a. \end{aligned}$$

b) We directly compute this expression using our results from the last subtask. We get

$$\frac{1}{\mathsf{det}(A)} C^\top = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Observe that this is exactly the formula for inverses of 2×2 matrices.

5. **a)** Using the determinant formula for 2×2 matrices, we directly obtain

$$\mathsf{det}(A) = \mathsf{det} \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a^2.$$

b) There are of course various ways to calculate determinants. One way is to use Proposition 5.1.13 (expansion along the first row) to get

$$\mathsf{det} B = b \cdot (-1)^3 \cdot \mathsf{det}(\begin{bmatrix} -b & 2 \\ 1 & 0 \end{bmatrix}) + (-1) \cdot (-1)^4 \cdot \mathsf{det}(\begin{bmatrix} -b & 0 \\ 1 & -2 \end{bmatrix}) = 2b - 2b = 0.$$

c) Let A be an arbitrary $n \times n$ matrix. Proposition 5.1.19 states that the determinant is linear in each row. To illustrate how we can use this here, let $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^n$ denote the rows of A, i.e. we have

$$A = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_n^\top & — \end{bmatrix}.$$

Using Proposition 5.1.19 n times (once for each row), we hence calculate

$$\mathsf{det}(-A) = \mathsf{det}\begin{bmatrix} — & -\mathbf{a}_1^\top & — \\ \vdots & \vdots & \vdots \\ — & -\mathbf{a}_n^\top & — \end{bmatrix} = (-1)^n \mathsf{det}\begin{bmatrix} — & \mathbf{a}_1^\top & — \\ \vdots & \vdots & \vdots \\ — & \mathbf{a}_n^\top & — \end{bmatrix} = (-1)^n \mathsf{det}(A).$$

d) As it turns out, the determinant of C must be 0. To see this, observe that since C is skew-symmetric, we must have $\mathsf{det}(C) = \mathsf{det}(-C^\top) = \mathsf{det}(-C)$. But by the previous subtask, we also know $\mathsf{det}(-C) = (-1)^n \mathsf{det}(C)$. Putting both things together, we obtain $\mathsf{det}(C) = (-1)^n \mathsf{det}(C)$. For odd n, this implies that $\mathsf{det}(C)$ must be zero.

6. a) Let $\mathbf{x}$ denote the vector of x-coordinates $\mathbf{x} = \begin{bmatrix} p_{x,1} & \dots & p_{x,n} \end{bmatrix}^\top$ and let $\mathbf{y}$ denote the vector of y-coordinates $\mathbf{y} = \begin{bmatrix} p_{y,1} & \dots & p_{y,n} \end{bmatrix}^\top$. The smoothness property can be rewritten as

$$\begin{aligned} \mathbf{p}_j - \frac{1}{2}(\mathbf{p}_{j-1} + \mathbf{p}_{j+1}) &= \mathbf{0} \ \forall j \in \{2, \dots, n-1\} \\ \mathbf{p}_1 - \frac{1}{2}(\mathbf{p}_n + \mathbf{p}_2) &= \mathbf{0} \\ \mathbf{p}_n - \frac{1}{2}(\mathbf{p}_{n-1} + \mathbf{p}_1) &= \mathbf{0} \end{aligned}$$

which translates to $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ with

$$A = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

The matrix A can also be written as

$$A = I - \frac{1}{2}(T + E_{1,n}) - \frac{1}{2}(T + E_{1,n})^\mathsf{T}$$

where T is the matrix with ones on the first strict upper diagonal (i.e. the entries where the row coefficient i and column coefficient j satisfy $j = i+1$ for $i \in \{1, ..., n-1\}$) and zeroes everywhere else, and $E_{1,n}$ has a single non-zero entry in row 1 and column n that is equal to 1.

We also want to satisfy the constraints $\mathbf{p}_{j_s} = \mathbf{c}_s$ for all $s \in [k]$. Let $\mathbf{x}^c$ denote the vector of x-coordinates of the locations, i.e. $\mathbf{x}^c = \begin{bmatrix} c_{x,1} & \dots & c_{x,k} \end{bmatrix}^\top$ and let $\mathbf{y}^c$ denote the vector of y-coordinates $\mathbf{y}^c = \begin{bmatrix} c_{y,1} & \dots & c_{y,n} \end{bmatrix}^\top$. Then, the location constraints can be written as $B\mathbf{x} = \mathbf{x}^c$ and $B\mathbf{y} = \mathbf{y}^c$ where the matrix $B \in \mathbb{R}^{k \times n}$ is given by $B_{s,r} = \delta_{r,j_s}$ for all $s \in [k]$ and $r \in [n]$ (recall that the Kronecker-Delta δ_{r,j_s} is one if $r = j_s$ and zero otherwise). In other words, an entry $B_{s,r}$ is one whenever the vertex $\mathbf{p}_r$ should match location $\mathbf{c}_s$ according to the prescribed correspondence $\mathcal{C}$, and $B_{s,r}$ is zero otherwise.

The final systems of linear equations hence are

$$\left[\begin{array}{c} A \\ \hline B \end{array}\right] \mathbf{x} = \left[\begin{array}{c} \mathbf{0_n} \\ \hline \mathbf{x}^c \end{array}\right] \text{ and } \left[\begin{array}{c} A \\ \hline B \end{array}\right] \mathbf{y} = \left[\begin{array}{c} \mathbf{0_n} \\ \hline \mathbf{y}^c \end{array}\right]$$

where $\mathbf{0}_n$ denotes the n dimensional all-zero vector.

b) Let $S = \left[\begin{array}{c} A \\ \hline B \end{array}\right]$ denote the system matrix. Indeed, the system matrix is the same for both linear systems. Since $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{k \times n}$, the system matrix S is in $\mathbb{R}^{(n+k) \times n}$. This implies that S has rank at most n.

c) We are solving for the curve vertex positions in the least squares sense for the values $n = 6$, $k = 3$, $\mathcal{C} = \{j_1 = 1, j_2 = 3, j_3 = 5\}$ and

$$\mathbf{c}_1 = \begin{bmatrix} c_{x,1} \\ c_{y,1} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_2 = \begin{bmatrix} c_{x,2} \\ c_{y,2} \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$
$$\mathbf{c}_3 = \begin{bmatrix} c_{x,3} \\ c_{y,3} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

Our strategy is to first combine the two linear systems in one larger system and then solve this using the least squares method. Observe that the two systems

$$\begin{bmatrix} A \\ \hline B \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{0_n} \\ \hline \mathbf{x}^c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A \\ \hline B \end{bmatrix} \mathbf{y} = \begin{bmatrix} \mathbf{0_n} \\ \hline \mathbf{y}^c \end{bmatrix}$$

can be rewritten as

$$M \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} A & 0_{n,n} \\ B & 0_{k,n} \\ 0_{n,n} & A \\ 0_{k,n} & B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0_n} \\ \mathbf{x}^c \\ \mathbf{0_n} \\ \mathbf{y}^c \end{bmatrix}$$

where M is a $2(n + k) \times 2n$ matrix block matrix (meaning that we put it together from smaller matrices) and $0_{n,n}$ and $0_{k,n}$ are zero-matrices of corresponding dimensions.

The normal equations hence yield

$$M^\top M \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = M^\top \begin{bmatrix} \mathbf{0_n} \\ \mathbf{x}^c \\ \mathbf{0_n} \\ \mathbf{y}^c \end{bmatrix}.$$

Plugging in the values of this specific example for A, B, $\mathbf{x}^c$, and $\mathbf{y}^c$, we get

$$S = \begin{bmatrix} A \\ B \end{bmatrix} = \left[\begin{array}{cccccc} 1 & -1/2 & 0 & 0 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & -1/2 \\ -1/2 & 0 & 0 & 0 & -1/2 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right], \quad \begin{bmatrix} \mathbf{0} \\ \mathbf{x}^c \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 2 \\ 6 \\ 4 \end{array}\right], \quad \begin{bmatrix} \mathbf{0} \\ \mathbf{y}^c \end{bmatrix} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 2 \\ 2 \\ 0 \end{array}\right].$$

The exact final solution is (obtained by solving the normal equations with a computer)

$$\mathbf{x} = \begin{bmatrix} 76/29 & 4 & 156/29 & 148/29 & 4 & 84/29 \end{bmatrix}^\top \text{ and } \mathbf{y} = \begin{bmatrix} 52/29 & 60/29 & 52/29 & 28/29 & 12/29 & 28/29 \end{bmatrix}^\top.$$

A drawing of this solution is provided in Figure 1 below.

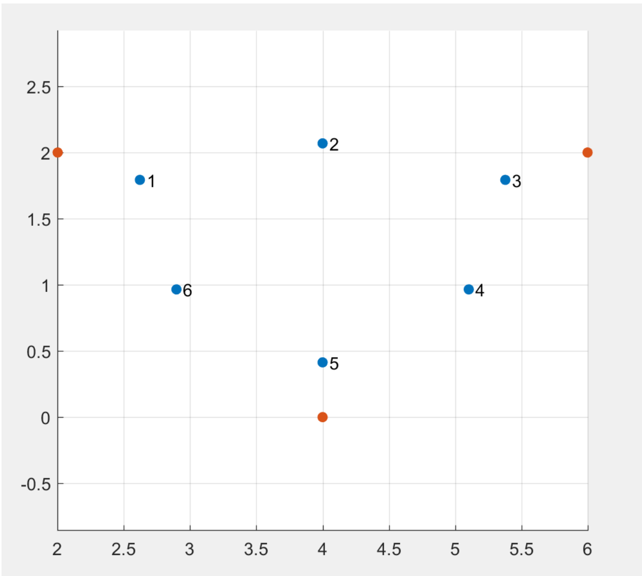

Figure 1: A drawing of the solution.