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Solution for Assignment 11

1. **a)** Consider the rotation Φ. Applying Φ to a vector should leave the second coordinate unchanged but rotate in the plane given by the first and third coordinate. In particular, deleting the second row and second column should give us the corresponding rotation matrix in two dimensions. Hence, the matrix A is given by

$$A = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 & \sqrt{3} \\ 0 & 2 & 0 \\ -\sqrt{3} & 0 & 1 \end{bmatrix}.$$

Note that this solution assumes that the direction of the rotation is given by the right-hand rule in a right-handed coordinate system, as explained e.g. on Wikipedia[1]. This was not properly specified in the original task description.

b) The vector $\begin{bmatrix}3 & 4 & 0\end{bmatrix}^\top \in \mathbb{R}^3$ is orthogonal to the plane P. Normalizing it yields the normal vector

$$\mathbf{n} = \begin{bmatrix}\frac{3}{5} & \frac{4}{5} & 0\end{bmatrix}^\top.$$

Now consider an arbitrary vector $\mathbf{x} \in \mathbb{R}^3$. We can split it into $\mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_|$ with $\mathbf{x}_\perp, \mathbf{x}_| \in \mathbb{R}^3$ such that $\mathbf{x}_\perp \cdot \mathbf{n} = 0$ and $\mathbf{x}_| \cdot \mathbf{n} = ||\mathbf{x}_|||$ (i.e. $\mathbf{x}_|$ is collinear with $\mathbf{n}$). In particular, we then must have $\mathbf{x}_\perp \in P$. Hence, reflecting $\mathbf{x}$ on the plane P should yield $\Psi(\mathbf{x}) = -\mathbf{x}_| + \mathbf{x}_\perp$. This works for any $\mathbf{x} \in \mathbb{R}^3$ and plugging in $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ for $\mathbf{x}$ will give you the desired matrix.

To avoid this computation, let us recall that we have previously projected vectors to a plane. In particular, the projection matrix here would be $I - \mathbf{n}\mathbf{n}^\top$. From that, one might now guess that the reflection matrix for reflection on the plane P should be $I - 2\mathbf{n}\mathbf{n}^\top$ (the term $-\mathbf{n}\mathbf{n}^\top$ in the projection gets rid of the $\mathbf{x}_|$ part, hence the term $-2\mathbf{n}\mathbf{n}^\top$ should negate the $\mathbf{x}_|$ part). Indeed, we can check that

$$(I - 2\mathbf{n}\mathbf{n}^\top)\mathbf{x} = (I - 2\mathbf{n}\mathbf{n}^\top)(\mathbf{x}_\perp + \mathbf{x}_|) = (\mathbf{x}_\perp + \mathbf{x}_|) - 2\mathbf{x}_| = \mathbf{x}_\perp - \mathbf{x}_|$$

as desired. Hence, we get

$$B = I - 2\mathbf{n}\mathbf{n}^\top = \frac{1}{25}\begin{bmatrix} 7 & -24 & 0 \\ -24 & -7 & 0 \\ 0 & 0 & 25 \end{bmatrix}.$$

c) Both matrices are square, hence it suffices to check $AA^\top = I$ and $BB^\top = I$. Indeed, we have

$$AA^\top = \frac{1}{4}\begin{bmatrix} 1+3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$BB^\top = \frac{1}{625}\begin{bmatrix} 7^2+24^2 & 0 & 0 \\ 0 & 24^2+7^2 & 0 \\ 0 & 0 & 25^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

d) As we have observed before, a rotation should leave the axis of rotation unchanged. In our case, the vector $\mathbf{e}_2 \in \mathbb{R}^3$ is on this axis of rotation. Indeed, we get $A\mathbf{e}_2 = \mathbf{e}_2$. Hence, $\mathbf{e}_2$ is a real eigenvector of A with corresponding eigenvalue 1. As it turns out, this is the only real eigenvalue of A and all other real eigenvectors of A are multiples of $\mathbf{e}_2$ (i.e. all vectors on the axis of rotation).

[1] https://en.wikipedia.org/wiki/Right-hand_rule

e) Consider the vector $\mathbf{n} \in \mathbb{R}^3$ from before. As we observed, we have $\Psi(\mathbf{n}) = B\mathbf{n} = -\mathbf{n}$ and hence $\mathbf{n}$ is a real eigenvector of B with corresponding eigenvalue -1. Similarly, vectors that lie in the plane P should not be affected by the reflection. Indeed, consider for example the vector $\mathbf{v} = \begin{bmatrix}-4 & 3 & 0\end{bmatrix}^\top$. We have $\mathbf{v} \cdot \mathbf{n} = 0$ and hence $\mathbf{v} \in P$ and $B\mathbf{v} = (I - 2\mathbf{n}\mathbf{n}^\top)\mathbf{v} = \mathbf{v}$. Hence, $\mathbf{v}$ is a real eigenvector of B with corresponding eigenvalue 1.

The matrix B does not have any other (distinct) eigenvalues because, as it turns out, the eigenvalue 1 appears with algebraic multiplicity two. (But it would be possible to find another real eigenvector corresponding to eigenvalue 1 that is linearly independent from $\mathbf{v}$.) This corresponds to the fact that all vectors in P (a two-dimensional object) are eigenvectors of B.

2. We compute

$$1 + \frac{1+\sqrt{5}}{2} = \frac{2+1+\sqrt{5}}{2} = \frac{1+5+2\sqrt{5}}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^2$$

and similarly

$$1 + \frac{1-\sqrt{5}}{2} = \frac{2+1-\sqrt{5}}{2} = \frac{1+5-2\sqrt{5}}{4} = \left(\frac{1-\sqrt{5}}{2}\right)^2.$$

3. **a)** Using the rules we learned in the lecture, we calculate

$$\begin{aligned}
u + v + w &= (u+v) + w = (4+2i) + (3-4i) = (4+3) + (2-4)i = 7 - 2i \\
u \cdot v &= (3+i)\cdot(1+i) = 3 + 3i + i - 1 = 2 + 4i \\
v \cdot w \cdot i &= (1+i)\cdot(3-4i)\cdot i = (3 - 4i + 3i + 4)\cdot i = 3i + 4 - 3 + 4i = 1 + 7i \\
\frac{w}{v} &= \frac{w}{v}\cdot\frac{\overline{v}}{\overline{v}} = \frac{(3-4i)(1-i)}{(1+i)(1-i)} = \frac{3-3i-4i-4}{1+1} = -\frac{1}{2} - \frac{7}{2}i \\
\frac{v}{u} &= \frac{v}{u}\cdot\frac{\overline{u}}{\overline{u}} = \frac{(1+i)(3-i)}{(3+i)(3-i)} = \frac{3-i+3i+1}{9+1} = \frac{2}{5} + \frac{1}{5}i \\
|v| &= \sqrt{1^2+1^2} = \sqrt{2}.
\end{aligned}$$

b) To find the polar form $re^{i\theta}$ of a complex number, first calculate its absolute value r. This can then be factored out and the corresponding angle θ can be determined with the help of trigonometry on the unit circle (and Euler's formula) to obtain

$$\begin{aligned}
3 &= 3(1+0i) = 3(\cos 0 + \sin 0 i) = 3e^{i0} \\
2i &= 2(0+i) = 2(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}i) = 2e^{i\frac{\pi}{2}} \\
1+\sqrt{3}i &= 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}i) = 2e^{i\frac{\pi}{3}} \\
5\sqrt{3} - 5i &= 10(\frac{\sqrt{3}}{2} - \frac{1}{2}i) = 10(\cos\frac{11\pi}{6} + \sin\frac{11\pi}{6}i) = 10e^{i\frac{11\pi}{6}}.
\end{aligned}$$

In case you are struggling to compute the angle θ, here are some details for the computation in the third calculation above. In particular, assume that we already found the radius and factored

$$1 + \sqrt{3}i = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i).$$

We now want to find $\theta \in [0, 2\pi)$ such that $\cos(\theta) = \frac{1}{2}$ and $\sin(\theta) = \frac{\sqrt{3}}{2}$. You can either look this up in a table or alternatively, in this case you can find the angle with the idea described in Figure 1 below.

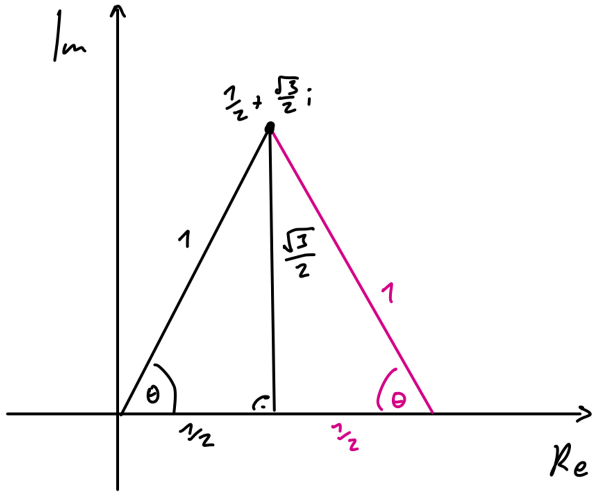

Figure 1: Assume we want to compute the angle of the complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$. In the drawing, the initial situation is drawn in black and the angle is labeled θ. For this particular complex number, it can be seen that θ must be $\pi/3$: we add a mirrored copy of the black triangle in pink. Together, the two triangles form a big triangle with three sides of length 1 each. Hence, all angles in this big triangle must be $\pi/3$.

c) We can also use Euler's formula to get from polar coordinates to cartesian coordinates:

$$-2e^{i\frac{\pi}{4}} = -2(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}i) = -2(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = -\sqrt{2} - \sqrt{2}i$$
$$4e^{i\frac{2\pi}{3}} = 4(\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}i) = 4(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) = -2 + 2\sqrt{3}i.$$

4. **a)** Let $z = x + iy \in \mathbb{C}$ be arbitrary. We want to understand the term $\Im(i\bar{z})$. A brief calculation reveals

$$\Im(i\bar{z}) = \Im(i(x - iy)) = \Im(ix + y) = x.$$

Hence, we have

$$1 < \Im(i\bar{z}) < 2 \iff 1 < x < 2$$

and we get the following drawing of A:

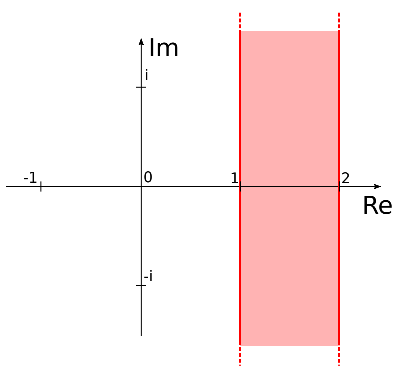

Figure 2: The set A is shown in red. Note that the two vertical lines through 1 and 2, respectively, are not part of A.

b) Let $z = x + iy \in \mathbb{C}$ be arbitrary. We simplify the condition in the definition of B as follows:

$$\begin{aligned} |z-2| < |z-2i| &\iff |z-2|^2 < |z-2i|^2 \\ &\iff (x-2)^2 + y^2 < x^2 + (y-2)^2 \\ &\iff x^2 - 4x + 4 + y^2 < x^2 + y^2 - 4y + 4 \\ &\iff x > y. \end{aligned}$$

Hence, we get the following drawing of B:

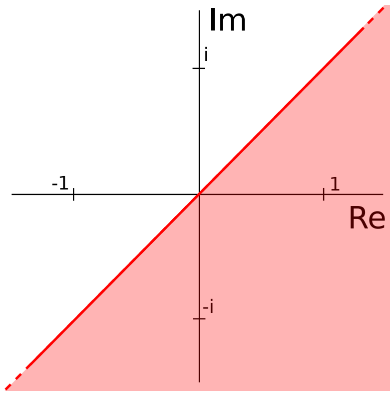

Figure 3: The set B is shown in red. Note that the diagonal line through 0 is not part of B.

c) Let $z = x + iy \in \mathbb{C}$ be arbitrary. We obtain the following simplification of the condition in C:

$$\begin{aligned} (\bar{z}+1)(z+1) = 2 \cdot \Im(z) &\iff |z|^2 + \bar{z} + z + 1 = 2 \cdot \Im(z) \\ &\iff x^2 + y^2 + x - iy + x + iy + 1 = 2y \\ &\iff x^2 + 2x + 1 + y^2 - 2y + (1-1) = 0 \\ &\iff (x+1)^2 + (y-1)^2 = 1. \end{aligned}$$

This is the equation of a circle with center at $x = -1, y = 1$ and radius 1. Hence, we get the following drawing of C:

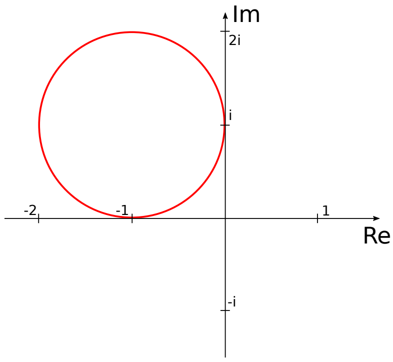

Figure 4: The set C is shown by the red circle with centerpoint $-1 + i$ and radius 1.

5. **a)** Note that 0 is an eigenvalue of A. In particular, the nullspace of A is non-trivial (i.e. contains vectors other than $\mathbf{0}$). Hence, A is not full rank and not invertible. By Proposition 5.1.2, this means that $\det(A) = 0$.

b) The given vector is a multiple of $\mathbf{v}_3$. In particular, it is the vector $18\mathbf{v}_3$. Thus, we have $A(18\mathbf{v}_3) = 18(A\mathbf{v}_3) = \mathbf{0}$ which means that the vector belongs to the nullspace of A.

c) Let V be the 3×3 matrix with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since the three vectors are orthonormal, V must be an orthogonal matrix, i.e. we have $VV^\top = V^\top V = I$.

We want to find the matrix A just from the information about its eigenvectors and eigenvalues. In particular, we know that $A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = -\mathbf{v}_2$, and $A\mathbf{v}_3 = \mathbf{0}$. We can summarize this in the equation

$$AV = VD$$

where $D \in \mathbb{R}^{3\times3}$ is the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiplying from the right by $V^{-1} = V^\top$ yields

$$A = VDV^\top.$$

Hence, we can now calculate A as follows:

$$\begin{aligned} A &= VDV^\top \\ &= \frac{1}{81}\begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 8 & -4 \\ -4 & 4 & 7 \\ 8 & 1 & 4 \end{pmatrix} \\ &= \frac{1}{81}\begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{81}\begin{pmatrix} -15 & 24 & 24 \\ 24 & 48 & -60 \\ 24 & -60 & -33 \end{pmatrix}. \end{aligned}$$

6. **a)** Let $\lambda \in \mathbb{R}$ be an arbitrary real eigenvalue of M with corresponding real eigenvector $\mathbf{v} \in \mathbb{R}^n$, i.e. we have

$$M\mathbf{v} = \lambda\mathbf{v}.$$

Now let's see what happens to $\mathbf{v}$ if we apply $M + cI$ instead of M to it:

$$\begin{aligned} (M + cI)\mathbf{v} &= M\mathbf{v} + c\mathbf{v} \\ &= \lambda\mathbf{v} + c\mathbf{v} \\ &= (\lambda + c)\mathbf{v}. \end{aligned}$$

As we have observed, $\mathbf{v}$ is a real eigenvector of $M + cI$ with corresponding real eigenvalue $c + \lambda$. This is exactly what we wanted to prove.

b) Consider the matrix

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}.$$

We observe that $A = B + 2I$. Hence, our plan is to find two distinct real eigenvalues of B, and then use the result from the previous subtask.

Since all rows of B are equal, the matrix has rank 1. Thus, 0 is an eigenvalue of B. It remains to find another real eigenvalue. For this, let us try to guess a real eigenvector of B that does not correspond to eigenvalue 0. This is not as hard as it may sound: every row of B is the same, hence any eigenvector of B that does not correspond to eigenvalue 0 should have the same value in each coordinate. Indeed, we have

$$B \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 36 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the vector $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$ is an eigenvector of B with corresponding eigenvalue 36.

By the result from the previous subtask, it follows that $\lambda_1 = 2$ and $\lambda_2 = 38$ are two distinct real eigenvalues of A.

c) Notice that $\mathbf{N}(A - \lambda I)$ is the subspace of eigenvectors corresponding to eigenvalue λ. We observe that

$$\mathbf{N}(A - \lambda I) = \mathbf{N}(B - (\lambda - 2)I)$$

since we have $A = B + 2I$.

For $\lambda_1 = 2$, we hence want to find the dimension of $\mathbf{N}(B)$. We already observed that the rank of B is one. Therefore, $\mathbf{N}(B)$ is of dimension $6 - 1 = 5$.

For $\lambda_2 = 38$, we want to find the dimension of $\mathbf{N}(A - \lambda_2 I) = \mathbf{N}(B - (\lambda_2 - 2)I) = \mathbf{C}(B^\top - (\lambda_2 - 2)I)^\perp$. Observe that $B^\top$ is a matrix of rank 1 where all columns are the same. Moreover, observe that we already found one eigenvector $\mathbf{1} \in \mathbb{R}^6$ corresponding to λ_2, hence the dimension of $\mathbf{N}(A - \lambda_2 I)$ is at least 1. We will find 5 linearly independent vectors in $\mathbf{C}(B^\top - (\lambda_2 - 2)I)$ which proves that actually, we have $\dim(\mathbf{N}(A - \lambda_2 I)) = 1$.

Concretely, consider the five vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is not hard to see that they are linearly independent since each of them fully controls one of the coordinates. Now define $\mathbf{w}_i = (B^\top - (\lambda_2 - 2)I)\left(\frac{-1}{\lambda_2 - 2}\mathbf{v}_i\right)$ for $i \in [5]$. For any $i \in [5]$, we get $B^\top \mathbf{v}_i = \mathbf{0}$ and hence

$$\mathbf{w}_i = (B^\top - (\lambda_2 - 2)I)\left(\frac{-1}{\lambda_2 - 2}\mathbf{v}_i\right) = \mathbf{v}_i.$$

In particular, we conclude that the vectors $\mathbf{w}_1 (= \mathbf{v}_1), \ldots, \mathbf{w}_5 (= \mathbf{v}_5)$ are all in $\mathbf{C}(B^\top - (\lambda_2 - 2)I)$ and linearly independent. Hence, we must have $\dim\left(\mathbf{C}(B^\top - (\lambda_2 - 2)I)\right) \geq 5$, as desired.

Of course, one could also solve this exercise differently. In particular, one could just compute the nullspaces using standard techniques.

7. Note that main idea behind solving these equations is to use the polar form.

a) Let us first simplify the equation a bit by writing

$$\begin{aligned} 3z^3 + 81 = 0 &\iff 3z^3 = -81 \\ &\iff z^3 = -27. \end{aligned}$$

Next, consider the polar form of $z \in \mathbb{C}$, i.e. write $z = re^{i\theta}$. Note that the polar form of the complex number -27 is $-27 = 27e^{i\pi}$. Substituting this yields

$$z^3 = -27 \iff r^3 e^{3i\theta} = 27e^{i\pi}.$$

We conclude that $r = \sqrt[3]{27} = 3$ and it remains to find the values of θ such that $e^{3i\theta} = e^{i\pi}$. Notice that adding multiples of $2\pi i$ in the exponent of the polar form does not change the complex number itself (adding $2\pi i$ corresponds to walking once around the unit circle). Hence, we want to find values for θ such that $3\theta = \pi + 2\pi k$ for some $k \in \mathbb{Z}$. Rearranging this yields

$$\theta = \frac{\pi}{3} + \frac{2\pi}{3}k$$

for $k \in \mathbb{Z}$. Technically, this means that there are infinitely many possibilities for θ (one for each $k \in \mathbb{Z}$). However, we do not actually need to consider the cases $k \geq 3$ and $k < 0$ (we restrict ourselves to angles in the interval $[0, 2\pi)$, all other solutions are redundant). Hence, we get

$$\theta_1 = \frac{\pi}{3}, \quad \theta_2 = \pi, \quad \theta_3 = \frac{5\pi}{3}$$

and therefore the three solutions

$$z_1 = 3e^{i\frac{\pi}{3}}, \quad z_2 = 3e^{i\pi}, \quad z_2 = 3e^{i\frac{5\pi}{6}}.$$

b) We proceed as in the previous subtask and rewrite $2z^2 + 4i = 0$ as $z^2 = -2i$. Substituting the polar form yields

$$z^2 = -2i \iff r^2 e^{i2\theta} = 2e^{i\frac{3\pi}{2}}.$$

From this, we get $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4} + \pi k$ for $k \in \mathbb{Z}$. Again, we can disregard the cases $k \geq 2$ and $k < 0$. Hence, we get the two solutions

$$z_1 = \sqrt{2}e^{i\frac{3\pi}{4}}, \quad z_2 = \sqrt{2}e^{i\frac{7\pi}{4}}.$$

c) We first rewrite the equation as follows:

$$\begin{aligned} z^2 - \sqrt{2}\left(2 - i^3 + e^{i\pi}\right) &= 0 \\ z^2 - \sqrt{2}\left(2 + i - 1\right) &= 0 \\ z^2 - \sqrt{2}\left(1 + i\right) &= 0 \\ z^2 &= \sqrt{2}\left(1 + i\right). \end{aligned}$$

To substitute the polar form, notice that $\sqrt{2}\,(1+i) = 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 2e^{i\frac{\pi}{4}}$. Hence, we get the equation

$$r^2 e^{i2\theta} = 2e^{i\frac{\pi}{4}}$$

in polar form. From this, we conclude $r = \sqrt{2}$ and $\theta = \frac{\pi}{8} + \pi k$ for $k \in \mathbb{Z}$. Again, the cases $k \geq 2$ and $k < 0$ are redundant and we get the two solutions

$$z_1 = \sqrt{2}e^{i\frac{\pi}{8}}, \quad z_2 = \sqrt{2}e^{i\frac{9\pi}{8}}.$$

8. By definition of the sequence, we get the relation

$$\begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

for all $n \geq 2$. We proceed analogously to the Fibonacci-example from the lecture. In particular, we start by finding the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial $p(\lambda)$ of A is given by

$$p(\lambda) = \begin{vmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

and hence has roots $\lambda_1 = 4$ and $\lambda_2 = -1$. By solving the linear systems

$$(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$$

and

$$(A - \lambda_2 I)\mathbf{x} = \mathbf{0}$$

we find the two corresponding eigenvectors (you might also be able to guess them)

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Now define $\mathbf{a}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$ for $n \geq 1$ and observe that

$$\mathbf{a}_1 = \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{5}\mathbf{v}_1 - \frac{3}{5}\mathbf{v}_2.$$

Hence, we get

$$\mathbf{a}_n = A^{n-1}\mathbf{a}_1 = A^{n-1}\frac{2}{5}\mathbf{v}_1 - A^{n-1}\frac{3}{5}\mathbf{v}_2 = \frac{2}{5}\lambda_1^{n-1}\mathbf{v}_1 - \frac{3}{5}\lambda_2^{n-1}\mathbf{v}_2 = \begin{bmatrix} 4^{n-1}\frac{2}{5}4 - (-1)^{n-1}\frac{3}{5} \\ 4^{n-1}\frac{2}{5} + (-1)^{n-1}\frac{3}{5} \end{bmatrix}$$

for all $n \geq 1$. We conclude that we have

$$a_n = 4^{n-1}\frac{2}{5}4 - (-1)^{n-1}\frac{3}{5} = \frac{2}{5}4^n + \frac{3}{5}(-1)^n.$$

for all $n \in \mathbb{N}_0$ (it is easy to check that it actually holds for $n = 0$ as well).