Solution for Assignment 12

1. **a)** Let $\mathbf{v}$ be an eigenvector of AB corresponding to eigenvalue λ. Observe that

$$(BA)B\mathbf{v} = B(AB)\mathbf{v} = B\lambda\mathbf{v} = \lambda B\mathbf{v}.$$

Hence, if $B\mathbf{v} \neq \mathbf{0}$ then $B\mathbf{v}$ is an eigenvector of BA with corresponding eigenvalue λ. Otherwise, we must have $B\mathbf{v} = \mathbf{0}$ and hence $\lambda = 0$. But this implies that B is not full rank which also means that BA is not full rank. Thus, $\lambda = 0$ must be an eigenvalue of BA.

b) Let $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ be a complete set of real eigenvectors of AB. By subtask a) and the additional assumption that B is invertible, we know that the vectors $B\mathbf{v}_1, \dots, B\mathbf{v}_n \in \mathbb{R}^n$ are all real eigenvectors of BA. Moreover, the n vectors $B\mathbf{v}_1, \dots, B\mathbf{v}_n$ are linearly independent since B is invertible and because the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent. Hence, $B\mathbf{v}_1, \dots, B\mathbf{v}_n$ form a basis of $\mathbb{R}^n$ and therefore they are a complete set of real eigenvectors of BA.

c) From subtask b), we know that if AB has a complete set of real eigenvectors, then so does BA. Using subtask b) again with matrices A and B exchanged (we can do this because here A is invertible as well), we also get that if BA has a complete set of real eigenvectors, then so does AB. This proves the claim.

d) Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The matrix BA has a complete set of real eigenvectors because $\mathbf{N}(BA) = \mathbb{R}^2$. However, the matrix AB does not have a complete set of real eigenvectors: the eigenvalue 0 appears with algebraic multiplicity 2, but AB only has rank 1 and hence $\dim(\mathbf{N}(AB)) = 1$. In other words, the geometric multiplicity of eigenvalue 0 is only 1. We conclude that there is no complete set of real eigenvectors for AB.

2. **a)** Since $\lambda_1, \dots, \lambda_n$ are distinct, we know from Proposition 6.1.7 that any n corresponding real eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ of A are linearly independent. In other words, A has a complete set of real eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. By assumption, B also has this complete set of real eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Let V be the matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$. From Theorem 6.2.1 it follows that $A = V\Lambda_A V^{-1}$ for some diagonal matrix $\Lambda_A \in \mathbb{R}^{n\times n}$. Analogously, it follows that $B = V\Lambda_B V^{-1}$ for some diagonal matrix $\Lambda_B \in \mathbb{R}^{n\times n}$. Now observe that since both Λ_A and Λ_B are diagonal matrices, we have $\Lambda_A\Lambda_B = \Lambda_B\Lambda_A$. Thus, we get

$$\begin{aligned} AB &= (V\Lambda_A V^{-1})(V\Lambda_B V^{-1}) \\ &= V\Lambda_A(V^{-1}V)\Lambda_B V^{-1} \\ &= V\Lambda_A\Lambda_B V^{-1} \\ &= V\Lambda_B\Lambda_A V^{-1} \\ &= V\Lambda_B(V^{-1}V)\Lambda_A V^{-1} \\ &= (V\Lambda_A V^{-1})(V\Lambda_B V^{-1}) \\ &= BA \end{aligned}$$

which concludes the proof.

b) By similarity of A and B, there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ with $A = SBS^{-1}$. By similarity of B and C, there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ with $B = TCT^{-1}$. Hence, we get

$$A = SBS^{-1} = STCT^{-1}S^{-1} = PCP^{-1}$$

where $P = ST$ is invertible with inverse $P^{-1} = T^{-1}S^{-1}$. We conclude that A and C are similar.

c) Since A has n distinct real eigenvalues, it must have a complete set of real eigenvectors and hence it must be diagonalizable (by Proposition 6.1.7 and Theorem 6.2.1), i.e. there exists an invertible matrix $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ with $A = V\Lambda V^{-1}$. Analogously, B is diagonalizable with $B = W\Lambda W^{-1}$ for some invertible matrix $W \in \mathbb{R}^{n\times n}$. Note that we can assume that the diagonalization of both A and B use the same diagonal matrix Λ because A and B are assumed to have the same eigenvalues.

We observe that this means that A is similar to Λ and that Λ is similar to B. By using the statement from the previous subtask for A, Λ, B, it follows that A is similar to B.

d) Let $\lambda \in \mathbb{R}$ be an arbitrary real eigenvalue of A with corresponding eigenvector $\mathbf{v} \in \mathbb{R}^n$. By similarity of A and B, there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ with $B = SAS^{-1}$. Now consider the vector $\mathbf{w} = S\mathbf{v}$. We have

$$B\mathbf{w} = SAS^{-1}(S\mathbf{v}) = SA\mathbf{v} = \lambda S\mathbf{v} = \lambda\mathbf{w}$$

and hence $\mathbf{w}$ is a real eigenvector of B with corresponding real eigenvalue λ. Since λ was arbitrary, we conclude that every real eigenvalue of A is also a real eigenvalue of B.

By a symmetric argument (swapping the roles of A and B above) we get that every real eigenvalue of B is also a real eigenvalue of A. We conclude that A and B have the same set of real eigenvalues.

3. **a)** Consider the vector $\mathbf{u} = \mathbf{v} + \mathbf{w}$. By linear independence of $\mathbf{v}$ and $\mathbf{w}$ we get that $\mathbf{u} \neq \mathbf{0}$. We also get

$$A\mathbf{u} = A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = 3\mathbf{w} + 3\mathbf{v} = 3\mathbf{u}$$

and hence 3 is an eigenvalue of A.

Analogously, we define $\mathbf{u}' = \mathbf{v} - \mathbf{w}$. By linear independence of $\mathbf{v}$ and $\mathbf{w}$ we get that $\mathbf{u}' \neq \mathbf{0}$. Again, we compute

$$A\mathbf{u}' = A(\mathbf{v} - \mathbf{w}) = A\mathbf{v} - A\mathbf{w} = 3\mathbf{w} - 3\mathbf{v} = -3\mathbf{u}$$

and hence -3 is also an eigenvalue of A.

b) First of all, note that the proof that we used in the previous subtask breaks down because without linear independence we cannot conclude that $\mathbf{u}$ and $\mathbf{u}'$ are both non-zero. The key insight for this subtask is that we can still guarantee that at least one of them is non-zero. Concretely, define $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\mathbf{u}' = \mathbf{v} - \mathbf{w}$ as before and assume for a contradiction that $\mathbf{u} = \mathbf{u}' = \mathbf{0}$. This would imply

$$\mathbf{u} + \mathbf{u}' = 2\mathbf{v} = \mathbf{0}$$

which is a contradiction to $\mathbf{v} \neq \mathbf{0}$. Hence, either $\mathbf{u} \neq \mathbf{0}$ or $\mathbf{u}' \neq \mathbf{0}$ and by the arguments from a), either 3 or -3 is an eigenvalue of A.

4. The key insight of this exercise is to look at the real eigenvalues of A. For this, we first compute the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} -\lambda & 1 & 3 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{3} & -\lambda \end{vmatrix} = -\lambda^3 + \frac{1}{2}\lambda + \frac{1}{2}.$$

We can guess one of the roots of this polynomial to be 1. Hence, we obtain

$$p(\lambda) = (\lambda - 1)(-\lambda^2 - \lambda - \frac{1}{2})$$

and it turns out that the other roots of p are complex-valued. Hence, the only real eigenvalue of A is 1.

Recall that our goal is to find an initial population that yields a stable population over time. The idea here is that an eigenvector corresponding to eigenvalue 1 is a suitable choice. Such an eigenvector can be found in $\mathbf{N}(A - I)$ and one example would be to choose

$$\mathbf{v}_0 = \begin{bmatrix} 6 & 3 & 1 \end{bmatrix}^\top .$$

Indeed, we get

$$A\mathbf{v}_0 = \begin{bmatrix} 0 & 1 & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix},$$

and hence $\mathbf{v}_t = A^t \mathbf{v}_0 = \mathbf{v}_0$ for all $t \in \mathbb{N}_0$. We conclude that this choice of initial population yields a stable population.

5. **a)** One could solve this by first computing A and then computing its real eigenvalues. But in this case, it is not hard to guess eigenvectors of A. In particular, choosing $x = y = 1$ yields

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and hence 1 is an eigenvalue of A. Moreover, guessing $x = 1$ and $y = -1$ yields

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and hence -1 is an eigenvalue of A. We conclude that the two real eigenvalues of A are 1 and -1. Since A is a 2×2 matrix, there cannot be more thant 2 eigenvalues.

b) The simplest choice is the diagonal matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Indeed, we have $A\mathbf{e}_1 = \mathbf{0}$, $A\mathbf{e}_2 = \mathbf{e}_2$, and $A\mathbf{e}_3 = 2\mathbf{e}_3$. Hence, A has the desired eigenvalues. It does not have any other eigenvalues because a 3×3 matrix can have at most 3 eigenvalues.

c) Consider the matrix A from the previous subtask and the basis

$$\mathbf{v}_1 = \mathbf{e}_1 + \mathbf{e}_2, \quad \mathbf{v}_2 = \mathbf{e}_1 - \mathbf{e}_2, \quad \mathbf{v}_3 = \mathbf{e}_3$$

of $\mathbb{R}^3$. In particular, consider the matrix

$$V = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with inverse

$$V^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and define B as

$$B = VAV^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The matrices A and B are similar and hence have the same eigenvalues.