Solution for Assignment 3

1. **a)** We will use the elimination procedure on A in order to get the upper triangular matrix U and the lower triangular matrix L. First, we multiply A with

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

to get

$$E_{31}E_{21}A = \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 1 & -11 \end{bmatrix}.$$

We also note down the coefficients $\ell_{21} = \frac{1}{2}$ and $\ell_{31} = 1$ for L. Note that these are just the negated entries of E_{21} and E_{31}, respectively. Next, we multiply this with $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$ and get

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & -10 \end{bmatrix} =: U$$

which is upper triangular. We also write down $\ell_{32} = \frac{1}{2}$. From the lecture we know that we can obtain L as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}.$$

Indeed, checking

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 2 & -12 & 6 \\ 1 & -4 & 1 \\ 2 & -11 & -5 \end{bmatrix} = A$$

we conclude that this is a valid LU factorization of A.

b) Since L is lower triangular, we can start substituting from the top. In particular, writing out the equation gives

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 4 \\ 4 \\ 25 \end{bmatrix} = \mathbf{b}.$$

Hence, we get $y_1 = 4$, $y_2 = 4 - 2 = 2$, and $y_3 = 25 - 1 - 4 = 20$.

c) We first write down the system again as

$$U\mathbf{x} = \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 4 \\ 2 \\ 20 \end{bmatrix} = \mathbf{y}.$$

By using back substitution we obtain $x_3 = \frac{20}{-10} = -2$, $x_2 = \frac{2-4}{2} = -1$, and $x_1 = \frac{4-12+12}{2} = 2$.

d) Using the results from the previous two subtasks we get

$$A\mathbf{x} \overset{LU}{=} LU\mathbf{x} = L(U\mathbf{x}) \overset{\text{c)}}{=} L\mathbf{y} \overset{\text{b)}}{=} \mathbf{b}.$$

2. **a)** To solve this exercise, we could use the elimination procedure for each system and then read off the solution. But the procedure would never actually eliminate anything (only reorder rows). In particular, the solutions to the system can be read off without using elimination as every row of A has a unique non-zero element. We first consider the system

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1.$$

To get the 1 in $\mathbf{e}_1$, the third coordinate of $\mathbf{x}_1$ has to be 1. To get the two zeroes of $\mathbf{e}_1$, the other two coordinates of $\mathbf{x}_1$ have to be zero. Hence, we get $\mathbf{x}_1 = \mathbf{e}_3$. Analogously, we obtain $\mathbf{x}_2 = \mathbf{e}_1$ and $\mathbf{x}_3 = \mathbf{e}_2$ from the other two systems.

b) Recall that the inverse A^{-1} of the $n \times n$ matrix A, if it exists, is the unique matrix satisfying $AA^{-1} = I$. Now recall our three linear systems from above:

$$\begin{aligned} A\mathbf{x}_1 &= \mathbf{e}_1 \\ A\mathbf{x}_2 &= \mathbf{e}_2 \\ A\mathbf{x}_3 &= \mathbf{e}_3 \end{aligned}$$

and consider what happens when we arrange the vectors $(A\mathbf{x}_1), (A\mathbf{x}_2), (A\mathbf{x}_3)$ as columns in a new matrix

$$\begin{bmatrix} | & | & | \\ A\mathbf{x}_1 & A\mathbf{x}_2 & A\mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = I.$$

Notice how this corresponds to the column view of matrix multiplication, i.e. we have

$$A \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ A\mathbf{x}_1 & A\mathbf{x}_2 & A\mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = I.$$

From this we can conclude that the matrix

$$X = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix}$$

must be the inverse of A. In particular, we have

$$A^{-1} = X = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{e}_3 & \mathbf{e}_1 & \mathbf{e}_2 \\ | & | & | \end{bmatrix}.$$

c) We proceed as above by first solving the three systems $D\mathbf{y}_1 = \mathbf{e}_1$, $D\mathbf{y}_2 = \mathbf{e}_2$, and $D\mathbf{y}_3 = \mathbf{e}_3$. Again, the solutions can be read off directly as D has a unique non-zero entry in every row. In particular, we get the solutions

$$\mathbf{y}_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}, \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

As above, we conclude that the matrix

$$Y = \begin{bmatrix} | & | & | \\ \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \\ | & | & | \end{bmatrix}$$

is the inverse of D since we have

$$DY = D \begin{bmatrix} | & | & | \\ \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ D\mathbf{y}_1 & D\mathbf{y}_2 & D\mathbf{y}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ | & | & | \end{bmatrix} = I.$$

Concretely, we have $D^{-1} = Y$.

d) Using the same strategy as above, one could now also determine the inverse of B. But the idea of this exercise is to observe that there is a faster way. In particular, we observe that B can be obtained from A and D as $DA = B$. From the lecture we know that B^{-1} can now be computed as

$$B^{-1} = (DA)^{-1} = A^{-1}D^{-1} = XY = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & 2 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

3. **a)** Yes, the inverse of A^k exists and is given by $(A^{-1})^k$. We will argue by induction over k.

- **Property**: The inverse of A^k is given by $(A^{-1})^k$.
- **Base case**: For $k = 1$, the property is true because we are given that A^{-1} is the inverse of A.
- **Induction step**: Fix a natural number $1 \leq k$ and assume that the property is true for this k (induction hypothesis). We prove that the property is true for $k + 1$, i.e. we prove that the inverse of A^{k+1} is $(A^{-1})^{k+1}$. By the induction hypothesis, we have $A^k(A^{-1})^k = I$. Hence, from

$$A^{k+1}(A^{-1})^{k+1} = A^k AA^{-1}(A^{-1})^k = A^k(AA^{-1})(A^{-1})^k = A^k I(A^{-1})^k = A^k(A^{-1})^k \overset{\text{IH}}{=} I$$

we conclude that the property is indeed true for $k + 1$.

b) We prove this by contradiction. Assume for a contradiction that A has an inverse A^{-1}. Since A is nilpotent, we know from Assignment 2 that $AA^{-1} = I$ is nilpotent too, i.e. there exists some $k \in \mathbb{N}^+$ such that $I^k = (AA^{-1})^k = 0$. But we have $I^k = I \neq 0$ which is a contradiction.

c) We prove this with the following calculation:

$$A = AI = A(A^3) = A^4 = I.$$

d) Observe that it suffices to find a 2×2 matrix $A \neq I$ with $A^2 = I$: indeed, for even $k = 2\ell$ we then get $A^k = A^{2\ell} = (A^2)^\ell = I^\ell = I$ as well by using $A^2 = I$. Such a matrix is sometimes called self-inverse, since we have $A^{-1} = A$. We have seen some self-inverse matrices before in this course. In particular, the 2×2 rotation matrix for angle $\phi = \pi$ is self-inverse. Concretely, this is the matrix

$$A = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and we can check that indeed $A^2 = I$ but $A \neq I$. Moreover, for odd k we then have

$$A^k = AA^{k-1} = AI = A \neq I$$

by using that $k - 1$ must be even.

e) Again, we use rotation matrices. In particular, the rotation matrix with angle $\pi/2$ should satisfy this. The intuition is that whenever we apply this rotation matrix four times, we rotate one whole turn and hence nothing happens. For a complete solution, we check that this indeed works. For the matrix

$$A = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

we get

$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ A^3 &= A^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ A^4 &= A^3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Now consider an arbitrary $k > 4$. Then k can be split as $k = 4\ell + h$ where $\ell, h \in \mathbb{N}_0$ and $0 \leq h < 4$ and we get $A^k = (A^4)^\ell A^h = A^h$. We conclude that $A^k = I$ if and only if $h = 0$ which is the same as saying that $A^k = I$ if and only if $k \equiv_4 0$.

4. In order to prove that S' is a subset of a hyperplane of $\mathbb{R}^{n+1}$, we need to find a vector $\mathbf{d}' \in \mathbb{R}^{n+1}$ such that $\mathbf{w} \cdot \mathbf{d}' = 0$ for all $\mathbf{w} \in S'$. To achieve this, consider what we know about an arbitrary $\mathbf{w} \in S'$: It must be of the form

$$\mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ 1 \end{bmatrix}$$

for some $\mathbf{v} \in S$. So for the scalar product $\mathbf{w} \cdot \mathbf{d}'$ we get

$$\mathbf{w} \cdot \mathbf{d}' = w_1 d_1' + w_2 d_2' + \cdots + w_{n+1} d_{n+1}' = v_1 d_1' + v_2 d_2' + \cdots + v_n d_n' + d_{n+1}'.$$

By $\mathbf{v} \in S$ we know that $\mathbf{v} \cdot \mathbf{d} = c$. So if we now choose

$$\mathbf{d}' = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \\ -c \end{bmatrix}$$

we get

$$\mathbf{w} \cdot \mathbf{d}' = v_1 d_1' + v_2 d_2' + \cdots + v_n d_n' + d_{n+1}' = (\mathbf{v} \cdot \mathbf{d}) - c = (\mathbf{v} \cdot \mathbf{d} - c) = 0$$

as desired. This works for any $\mathbf{w} \in S'$ and hence S' is a subset of the hyperplane $\{\mathbf{v} \in \mathbb{R}^{n+1} : \mathbf{v} \cdot \mathbf{d}'\}$.

5. **a)** Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary and assume $B\mathbf{x} = \mathbf{0}$. Recall that $B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_n\mathbf{b}_n$ where $x_1, x_2, \ldots, x_n$ are the entries of $\mathbf{x}$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ are the columns of B. Hence, in order to prove that the columns of B are linearly independent, we have to prove $\mathbf{x} = \mathbf{0}$.

We achieve this with the calculation

$$\mathbf{x} = I\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}.$$

b) Let $\mathbf{y} \in \mathbb{R}^n$ be arbitrary and assume $A\mathbf{y} = \mathbf{0}$. We want to prove that $\mathbf{y} = \mathbf{0}$. By the inverse theorem and using subtask a), we know that there exists $\mathbf{x} \in \mathbb{R}^n$ with $B\mathbf{x} = \mathbf{y}$. Now observe that indeed, we have

$$\mathbf{y} = B\mathbf{x} = BI\mathbf{x} = B(AB\mathbf{x}) = B(A\mathbf{y}) = \mathbf{0}$$

and hence we conclude that the columns of A are linearly independent.

c) By applying the matrix A to $BA - I$ we get

$$A(BA - I) = ABA - A = IA - A = 0.$$

In particular, we have $A\mathbf{v} = \mathbf{0}$ for every column $\mathbf{v}$ of $BA - I$. But by the inverse theorem and subtask b), the equation $A\mathbf{y} = \mathbf{0}$ has a unique solution $\mathbf{y}$. And we also know that $\mathbf{0}$ is a solution since $A\mathbf{0} = \mathbf{0}$. By uniqueness, we conclude that we have $\mathbf{v} = \mathbf{0}$ for every column $\mathbf{v}$ of $BA - I$ and hence $BA - I = 0$.