Solution for Assignment 5

1. **a)** The key insight here is to use $\mathbf{v}^\top\mathbf{v} = \mathbf{v}\cdot\mathbf{v} = \|\,\mathbf{v}\,\|^2 = 1$. We then get

$$A^2 = (\mathbf{v}\mathbf{v}^\top)(\mathbf{v}\mathbf{v}^\top) = \mathbf{v}(\mathbf{v}^\top\mathbf{v})\mathbf{v}^\top = \mathbf{v}1\mathbf{v}^\top = \mathbf{v}\mathbf{v}^\top = A$$

for A^2 and therefore

$$P^2 = (I-A)^2 = I^2 - 2A + A^2 = I - 2A + A = I - A = P$$

for P^2.

b) Knowing that $\mathbf{w}\cdot\mathbf{v} = 0$, we compute $A\mathbf{w} = (\mathbf{v}\mathbf{v}^\top)\mathbf{w} = \mathbf{v}(\mathbf{v}^\top\mathbf{w}) = \mathbf{v}(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{v} = 0\mathbf{v} = \mathbf{0}$.

c) We are given $A\mathbf{w} = \mathbf{0}$ and hence get $\mathbf{0} = A\mathbf{w} = (\mathbf{v}\mathbf{v}^\top)\mathbf{w} = \mathbf{v}(\mathbf{v}^\top\mathbf{w}) = \mathbf{v}(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{v}$. Note that $\mathbf{v}\cdot\mathbf{w}$ is a scalar and that $\mathbf{v} \neq \mathbf{0}$. From $(\mathbf{v}\cdot\mathbf{w})\mathbf{v} = \mathbf{0}$ we hence get $\mathbf{v}\cdot\mathbf{w} = 0$, as desired.

d) Combining subtasks b) and c), we get that for all $\mathbf{w} \in \mathbb{R}^3$, we have $A\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v}\cdot\mathbf{w} = 0$. Hence, the nullspace of A is given by $\mathbf{N}(A) = \{\mathbf{w} \in \mathbb{R}^3 : \mathbf{w}\cdot\mathbf{v} = 0\}$. In words, this is the set of all vectors orthogonal to $\mathbf{v}$.

e) Observe that A has the form

$$A = \begin{bmatrix} | & | & | \\ v_1\mathbf{v} & v_2\mathbf{v} & v_3\mathbf{v} \\ | & | & | \end{bmatrix}$$

where $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$. Hence, all of its columns are linearly dependent on the first one. By $\mathbf{v} \neq \mathbf{0}$, we conclude that A has rank 1. Therefore, it is not invertible. Note that we have studied matrices of rank 1 before in Exercise 3 of Assignment 2.

f) We first prove $\mathbf{C}(A) \subseteq \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$. Let $\mathbf{w} \in \mathbf{C}(A)$ be arbitrary. By definition of $\mathbf{C}(A)$, there exists $\mathbf{x}$ with $A\mathbf{x} = \mathbf{w}$. We conclude $\mathbf{w} = A\mathbf{x} = \mathbf{v}\mathbf{v}^\top\mathbf{x} = (\mathbf{v}\cdot\mathbf{x})\mathbf{v} = \alpha\mathbf{v}$ for $\alpha = \mathbf{v}\cdot\mathbf{x}$. Thus, $\mathbf{w} \in \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$.

It remains to prove $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} \subseteq \mathbf{C}(A)$. Let $\mathbf{w} \in \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$ be arbitrary, i.e. $\mathbf{w} = \alpha\mathbf{v}$ for some $\alpha \in \mathbb{R}$. Choosing $\mathbf{x} = \mathbf{w}$, we get

$$A\mathbf{x} = \mathbf{v}\mathbf{v}^\top\mathbf{x} = \mathbf{v}\mathbf{v}^\top(\alpha\mathbf{v}) = \alpha(\mathbf{v}\cdot\mathbf{v})\mathbf{v} = \alpha\mathbf{v} = \mathbf{w}$$

and therefore $\mathbf{w} \in \mathbf{C}(A)$.

g) By subtask f), it suffices to prove $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$. To see that $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} \subseteq \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$, observe that $A(\alpha\mathbf{v}) = \alpha\mathbf{v}$ (we have calculated this in more detail already above) and hence $\alpha\mathbf{v} \in \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$ for all α. It remains to prove $\{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\} \subseteq \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$. By definition, any $\mathbf{w} \in \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$ is also in $\mathbf{C}(A)$. Hence, $\{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\} \subseteq \mathbf{C}(A)$ and with subtask f) we conclude the proof.

h) Let $\mathbf{w} \in \mathbb{R}^3$ be arbitrary. We have

$$P\mathbf{w} = \mathbf{0} \iff (I-A)\mathbf{w} = \mathbf{0} \iff \mathbf{w} - A\mathbf{w} = \mathbf{0} \iff A\mathbf{w} = \mathbf{w}.$$

This implies $\mathbf{N}(P) = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$ and by subtask g) we conclude $\mathbf{N}(P) = \mathbf{C}(A)$.

i) We start by proving $\mathbf{C}(P) \subseteq \mathbf{N}(A)$. Let $\mathbf{w} \in \mathbf{C}(P)$ be arbitrary. By definition, there is $\mathbf{x} \in \mathbb{R}^3$ with $P\mathbf{x} = \mathbf{w}$. We use this to compute $A\mathbf{w}$ as

$$A\mathbf{w} = A(P\mathbf{x}) = A(I - A)\mathbf{x} = (A - A^2)\mathbf{x} = (A - A)\mathbf{x} = \mathbf{0}$$

and conclude $\mathbf{w} \in \mathbf{N}(A)$.

It remains to prove $\mathbf{N}(A) \subseteq \mathbf{C}(P)$. Let $\mathbf{w} \in \mathbf{N}(A)$ be arbitrary. By definition, we have $A\mathbf{w} = \mathbf{0}$. Choosing $\mathbf{x} = \mathbf{w}$, we get $P\mathbf{x} = (I - A)\mathbf{x} = \mathbf{x} - A\mathbf{x} = \mathbf{w} - A\mathbf{w} = \mathbf{w}$. We conclude $\mathbf{w} \in \mathbf{C}(P)$.

2. a) The vector $\mathbf{b}$ is an element of U if it can be written as a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. This is equivalent to the existence of a solution of the linear system

$$\begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ | & | & | \end{bmatrix} \mathbf{x} = \mathbf{b}.$$

To solve the linear system, we need to do elimination on the matrix and also apply the same row operations to $\mathbf{b}$. It is sometimes convenient to do both at once by first concatenating $\mathbf{b}$ to the matrix (as a new column) and then applying elimination on this new matrix. By concatenating $\mathbf{b}$ to the matrix we get

$$\begin{bmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{b} \\ | & | & | & | \end{bmatrix}.$$

To remember that the last column comes from $\mathbf{b}$, we can insert a vertical separator (this is just notation) as follows:

$$\left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{b} \\ | & | & | & | \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ -4 & 5 & -5 & -2 \\ 8 & 5 & 5 & 6 \\ 2 & 2 & 1 & 2 \end{array}\right].$$

We get

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 9 & -3 & 2 \\ 0 & 3 & -1 & 1 \end{array}\right]$$

after performing elimination in the first column and finally

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

after performing elimination also in the second column. In particular, this means that the system

$$\begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ | & | & | \end{bmatrix} \mathbf{x} = \mathbf{b}.$$

has the same solutions as the system

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

But this last system does not have a solution and hence $\mathbf{b} \notin U$.

b) Interpreting the result of the elimination from the previous part of the task, we can determine that the rank of the system matrix is 2 (there are two pivots). Hence, the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent and therefore do not form a basis.

3. Each of the four points yields one linear equation with variables a, b, c, d. For example, for $x = 4, y = 5$ we get the equation

$$a4^3 + b4^2 + c4 + d = 5.$$

In total, we get the linear system

$$\begin{aligned} a0^3 + b0^2 + c0 + d &= 1 \\ a2^3 + b2^2 + c2 + d &= 2 \\ a4^3 + b4^2 + c4 + d &= 5 \\ a6^3 + b6^2 + c6 + d &= 6 \end{aligned}$$

with four equations and four variables that we can write in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 6 & 36 & 216 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}.$$

We now want to solve this system by using the elimination technique. For this, it is convenient to apply the row operations to the system matrix and the right-hand side simultaneously by appending the right-hand side to the matrix as follows:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 4 & 8 & 2 \\ 1 & 4 & 16 & 64 & 5 \\ 1 & 6 & 36 & 216 & 6 \end{array}\right].$$

After performing elimination in the first column we get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 4 & 16 & 64 & 4 \\ 0 & 6 & 36 & 216 & 5 \end{array}\right].$$

Next, we perform elimination in the second columns to get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 24 & 192 & 2 \end{array}\right].$$

Finally, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 0 & 48 & -4 \end{array}\right].$$

It remains to perform backward substitution. From the last row, we get $a = -\frac{4}{48} = -\frac{1}{12}$. Next, we get $b = \frac{2-48a}{8} = \frac{6}{8} = \frac{3}{4}$. From the second row we obtain $c = \frac{1-8a-4b}{2} = \frac{1+\frac{2}{3}-3}{2} = -\frac{2}{3}$. Finally, we get $d = 1$ from the first row. Hence, the function $f(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 1$ interpolates all of our datapoints.

4. **a)** Note that the function $\mathbf{0} : x \in \mathbb{R} \mapsto 0$ is both in O and E. Hence, both sets are non-empty. Thus, it remains to prove that both O and E are closed under vector addition and scalar multiplication. We start with O. Let $\mathbf{f}, \mathbf{g} \in O$ and $c \in \mathbb{R}$ be arbitrary. We have

$$(\mathbf{f} + \mathbf{g})(-x) = \mathbf{f}(-x) + \mathbf{g}(-x) = -\mathbf{f}(x) - \mathbf{g}(x) = -(\mathbf{g} + \mathbf{f})(x)$$

for all $x \in \mathbb{R}$ and hence $\mathbf{f} + \mathbf{g} \in O$. Similarly, we have

$$(c\mathbf{f})(-x) = c\mathbf{f}(-x) = -c\mathbf{f}(x) = -(c\mathbf{f})(x)$$

for all $x \in \mathbb{R}$ which proves $c\mathbf{f} \in O$. We conclude that O is a subspace of V.

We proceed analogously for E. Let $\mathbf{f}, \mathbf{g} \in E$ and $c \in \mathbb{R}$ be arbitrary. We have

$$(\mathbf{f} + \mathbf{g})(-x) = \mathbf{f}(-x) + \mathbf{g}(-x) = \mathbf{f}(x) + \mathbf{g}(x) = (\mathbf{g} + \mathbf{f})(x)$$

for all $x \in \mathbb{R}$ and hence $\mathbf{f} + \mathbf{g} \in E$. And also

$$(c\mathbf{f})(-x) = c\mathbf{f}(-x) = c\mathbf{f}(x) = (c\mathbf{f})(x)$$

for all $x \in \mathbb{R}$ which proves $c\mathbf{f} \in E$. We conclude that E is a subspace of V.

b) We already know that $\mathbf{0}$ is in both O and E and therefore $\mathbf{0} \in O \cap E$.

Now consider an arbitrary function $\mathbf{f} \in O \cap E$ and fix $x \in \mathbb{R}$. By definition of O, we have $\mathbf{f}(-x) = -\mathbf{f}(x)$. By definition of E, we also get $\mathbf{f}(-x) = \mathbf{f}(x)$. We conclude that we must have $-\mathbf{f}(x) = \mathbf{f}(x)$. But this implies $\mathbf{f}(x) = 0$. Since this works for any $x \in \mathbb{R}$, we conclude that $\mathbf{f}$ must be the zero function $\mathbf{0}$. Hence, $\mathbf{0}$ is the only function in $O \cap E$.

c) Let $\mathbf{f} \in V$ be arbitrary and define

$$\mathbf{g}(x) := \frac{1}{2}(\mathbf{f}(x) + \mathbf{f}(-x))$$
$$\mathbf{h}(x) := \frac{1}{2}(\mathbf{f}(x) - \mathbf{f}(-x))$$

for all $x \in \mathbb{R}$. Observe that we have $\mathbf{f} = \mathbf{g} + \mathbf{h}$. It remains to prove $\mathbf{g} \in E$ and $\mathbf{h} \in O$: For all $x \in \mathbb{R}$ we have

$$\mathbf{g}(-x) = \frac{1}{2}(\mathbf{f}(-x) + \mathbf{f}(-(-x))) = \frac{1}{2}(\mathbf{f}(x) + \mathbf{f}(-x)) = \mathbf{g}(x)$$

and hence $\mathbf{g} \in E$. Similarly, we have

$$\mathbf{h}(-x) = \frac{1}{2}(\mathbf{f}(-x) - \mathbf{f}(-(-x))) = \frac{1}{2}(\mathbf{f}(-x) - \mathbf{f}(x)) = -\frac{1}{2}(\mathbf{f}(x) - \mathbf{f}(-x)) = -\mathbf{h}(x)$$

for all $x \in \mathbb{R}$. Hence, $\mathbf{h} \in O$.

5. 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

Which of the following sets of vectors is a basis of $\mathbb{R}^4$?

(a)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A set of 5 vectors from $\mathbb{R}^4$ can never be linearly independent. Hence, this is not a basis.

(b)

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

The zero vector is linearly dependent on all other vectors. Hence, this is not a basis.

√ **(c)**

$$\left\{ \mathbf{v}_1, \quad \mathbf{v}_2, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

These 4 vectors are linearly independent. If we put them as columns into a matrix A, then A will have full rank. By the inverse theorem, the system $A\mathbf{x} = \mathbf{b}$ will have a unique solution for all $\mathbf{b} \in \mathbb{R}^4$. Thus, $\mathbf{C}(A) = \mathbb{R}^4$ or in other words, the four vectors span all of $\mathbb{R}^4$. Thus, they are a basis of $\mathbb{R}^4$.

2. Which of the following matrices are in reduced row echelon form?

(a) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Not in reduced row echolon form because of the 2 in the first row.

√ **(b)** $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in reduced row echolon form. There are two pivots, one in the first column and one in the second column.

√ **(c)** $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in reduced row echolon form. The pivots are in the first and third column.

(d) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Not in reduced row echolon form because the 1 and 2 in the first and second row of the third column have not been eliminated.