Linear Algebra, First Part
Blackboard Notes

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October 31, 2023

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Chapter 0

Preface

These are the blackboard notes for the first half of the course

Lineare Algebra (401-0131-00L)

held at the Department of Computer Science at ETH Zürich in HS23. The notes roughly correspond to what I plan to write on the tablet during my lectures (in German for the first half of the course). The actual tablet notes will be made available after each lecture.

In structure and content, the notes are based on the book

> *Introduction to Linear Algebra* (Sixth Edition) by Gilbert Strang, Wellesley - Cambridge Press, 2023.

The notes are rather dense and not meant to replace full lecture notes or a book. Mainly, they should free students from the need to copy material from the blackboard. Many additional explanations (and answers to questions) will be given in the lectures. Exercises to practice the material will be published in the course Moodle and are discussed during the exercise classes.

To summarize, these notes do not represent a complete and standalone Linear Algebra course; rather, they are meant to support the lectures and exercise classes.

I also want to point out that Strang's book is *not* part of the course's official material, and there is no need for students to buy the book. With the blackboard notes, exercises, lectures, and exercises classes, the course is self-contained. Strang's book serves as recommended but optional literature.

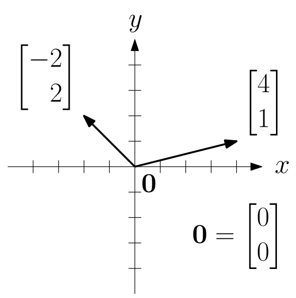
Bernd Gärtner, Zürich, September 5, 2023

Chapter 1

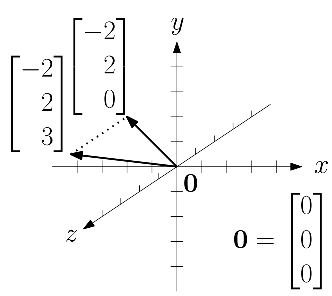
Vectors and Matrices

1.1 Vectors and Linear Combinations

A vector is (for now) an element of $\mathbb{R}^n$ vector = sequence (tuple) of n real numbers



$\mathbb{R}^2$: xy-plane



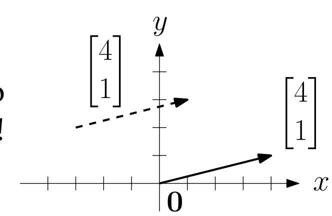
$\mathbb{R}^3$: xyz-space

$$\mathbf{u} = \begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}, \mathbf{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}, \mathbf{w} = \begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix}, \ldots$$

$\mathbb{R}^n$

$\mathbb{R}$: real numbers
$n \in \mathbb{N}$ (natural numbers)
$\mathbf{0}$: zero vector.

Vector = "movement" : go 4 steps right and 1 step up!

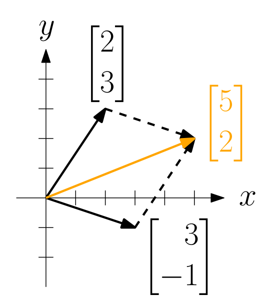


1.1.1 Vector addition: $\mathbf{v} + \mathbf{w}$

Combine the movements!

$$\mathbb{R}^2 : \begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}$$

$$\mathbb{R}^n : \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} + \begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix} = \begin{bmatrix}v_1+w_1\\v_2+w_2\\\vdots\\v_n+w_n\end{bmatrix}$$

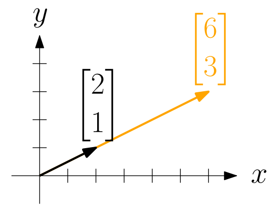


"Parallelogram"

1.1.2 Scalar multiplication: $c\mathbf{v}$

Move c times as far! (c: the scalar)

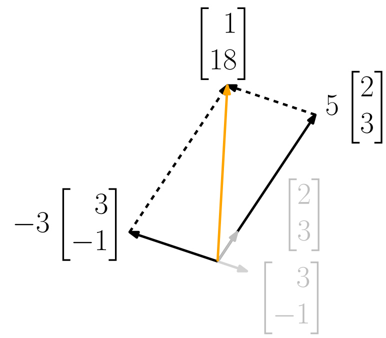
$$\mathbb{R}^2 : 3\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}6\\3\end{bmatrix} \quad \Bigg| \quad \mathbb{R}^n : c\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} = \begin{bmatrix}cv_1\\cv_2\\\vdots\\cv_n\end{bmatrix}$$



1.1.3 (Linear) combination: $c\mathbf{v} + d\mathbf{w}$

$$5\begin{bmatrix}2\\3\end{bmatrix} - 3\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}10\\15\end{bmatrix} - \begin{bmatrix}9\\-3\end{bmatrix} = \begin{bmatrix}1\\18\end{bmatrix}$$

Here: $c = 5, d = -3$.

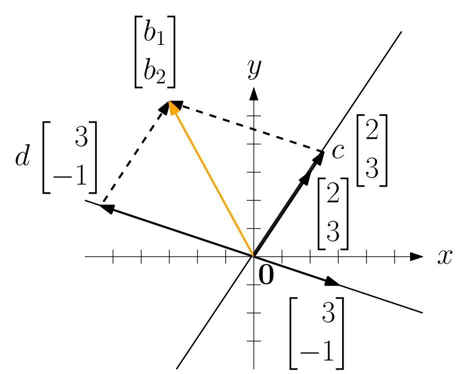


Every vector $\mathbf{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ is a combination of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\-1\end{bmatrix}$! *Proof:* we want c and d such that

$$c\begin{bmatrix}2\\3\end{bmatrix} + d\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}.$$

"Column Picture:"

Draw a parallelogram with opposite corners $\mathbf{0}$ and $\mathbf{b}$ and sides parallel to $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\-1\end{bmatrix}$. The other two corners are $c\begin{bmatrix}2\\3\end{bmatrix}$ and $d\begin{bmatrix}3\\-1\end{bmatrix}$.

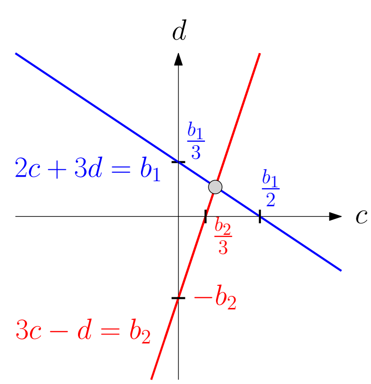


"Row picture:"

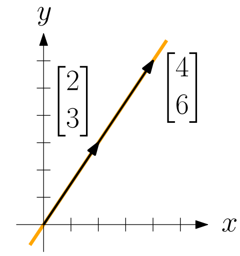
Two equations in two unknowns c and d:

$$\begin{aligned} 2c + 3d &= b_1 \\ 3c - d &= b_2 \end{aligned}$$

Draw them as lines in the cd-plane. The intersection point solves both equations.



Doesn't always work: All combinations of $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\6\end{bmatrix}$ are on a line! (Exercise: What goes wrong in column and row pictures?)



1.1.4 Combining more vectors, matrix notation

$$\underbrace{3\begin{bmatrix}1\\2\end{bmatrix}+2\begin{bmatrix}-1\\3\end{bmatrix}-4\begin{bmatrix}0\\1\end{bmatrix}}_{\text{combination of 3 vectors}}=\begin{bmatrix}1\\8\end{bmatrix}$$

$$\underbrace{\underbrace{\begin{bmatrix}1&-1&0\\2&3&1\end{bmatrix}}_{\text{matrix}}\begin{bmatrix}3\\2\\-4\end{bmatrix}}_{\text{matrix-vector multiplication}}=\begin{bmatrix}1\cdot 3-1\cdot 2-0\cdot 4\\2\cdot 3+3\cdot 2-1\cdot 4\end{bmatrix}=\begin{bmatrix}1\\8\end{bmatrix}$$

$$\underbrace{c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_n\mathbf{v}_n}_{\text{combination of } n \text{ vectors in } \mathbb{R}^m}=\mathbf{b}$$

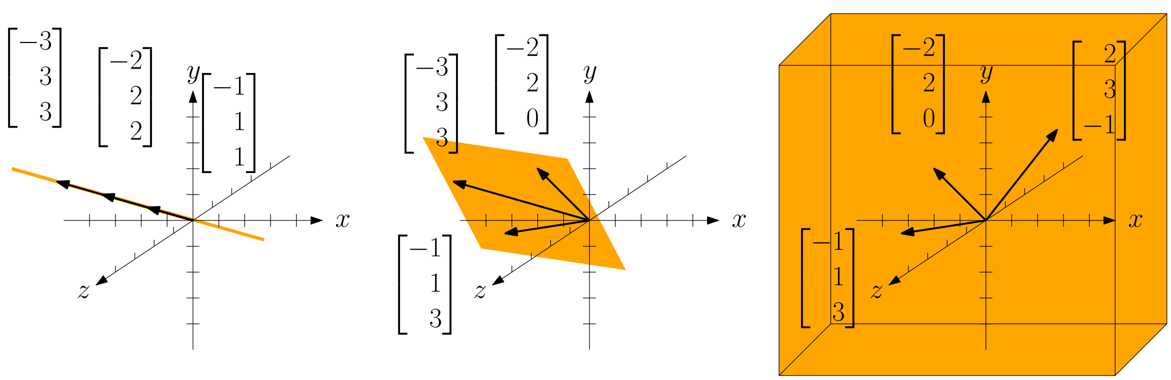
$$m \text{ rows}\quad \overbrace{\underbrace{\begin{bmatrix}|&|&&|\\\mathbf{v}_1&\mathbf{v}_2&\cdots&\mathbf{v}_n\\|&|&&|\end{bmatrix}}_{n\text{ columns}}}^{m\times n\text{ matrix}}\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}=\underset{\uparrow}{\begin{bmatrix}|\\\mathbf{b}\\|\end{bmatrix}}$$

Matrix: "container for vectors"

$m \times 1$ matrix: a single vector in $\mathbb{R}^m$

1.1.5 Three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^3$

The combinations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ form a line (vectors are *collinear*), a plane (vectors are *coplanar*), or the whole space (vectors are *independent*).



1.2 Lengths and Angles from Dot Products

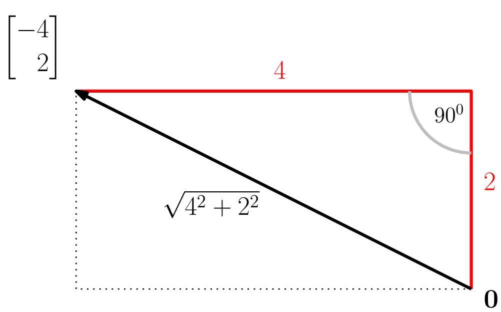
1.2.1 Scalar product (or dot product, inner product): $\mathbf{v} \cdot \mathbf{w}$

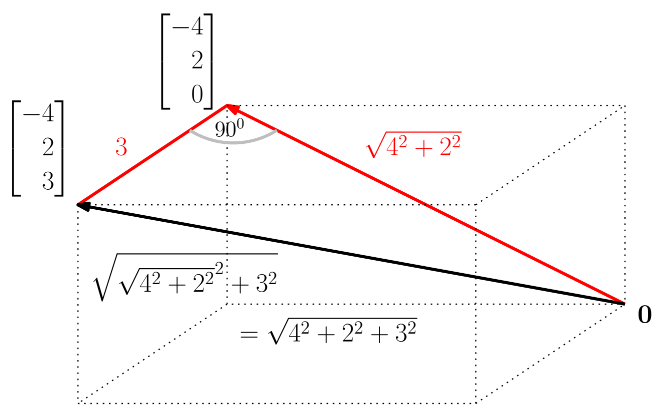
$$\mathbb{R}^2 : \begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}4\\6\end{bmatrix} = 1 \cdot 4 + 2 \cdot 6 = 16 \qquad \Bigg| \qquad \mathbb{R}^n : \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} \cdot \begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix} = v_1w_1 + v_2w_2 + \cdots + v_nw_n.$$

1.2.2 Length of a vector: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

$$\mathbb{R}^2 : \left\|\begin{bmatrix}-4\\2\end{bmatrix}\right\| = \sqrt{(-4)^2 + 2^2} = \sqrt{20} \qquad \Bigg| \qquad \mathbb{R}^n : \left\|\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}\right\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Why? Pythagoras!

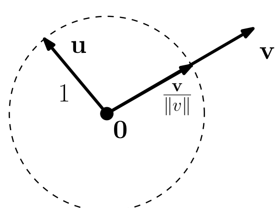




Unit vector: $\|\mathbf{u}\| = 1$.
For every $\mathbf{v} \neq \mathbf{0}$,

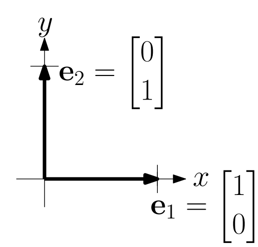
$$\frac{\mathbf{v}}{\|\mathbf{v}\|}$$

is a unit vector.



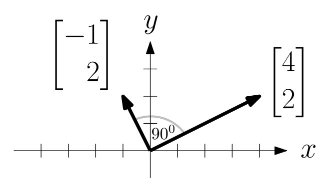
Standard unit vectors:

$$\mathbf{e}_i = \begin{bmatrix}0\\\vdots\\1\\\vdots\\0\end{bmatrix} \leftarrow \text{position } i$$

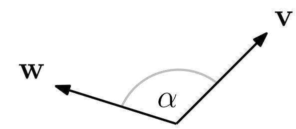


1.2.3 Perpendicular (or orthogonal) vectors: $\mathbf{v} \cdot \mathbf{w} = 0$

$$\begin{bmatrix}4\\2\end{bmatrix} \cdot \begin{bmatrix}-1\\2\end{bmatrix} = -4 \cdot 1 + 2 \cdot 2 = 0.$$



Cosine Formula:

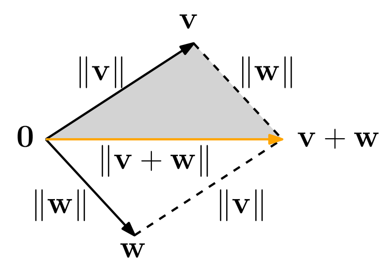


$$\cos(\alpha) = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \quad \text{for } \mathbf{v},\mathbf{w} \neq \mathbf{0}.$$

Because $|\cos(\alpha)| \leq 1$:

$$\text{Cauchy-Schwarz inequality:} \qquad \underbrace{|\mathbf{v}\cdot\mathbf{w}|}_{|\cos(\alpha)|\|\mathbf{v}\|\|\mathbf{w}\|} \quad \leq \quad \|\mathbf{v}\|\|\mathbf{w}\|.$$

Triangle inequality:



$$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

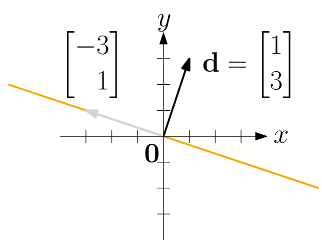
"From $\mathbf{0}$ directly to $\mathbf{v}+\mathbf{w}$ is shorter than via $\mathbf{v}$ or $\mathbf{w}$."

Hyperplanes.

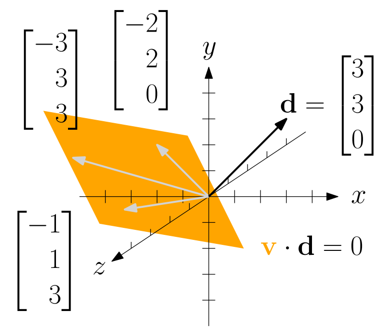
If $\mathbf{d} \in \mathbb{R}^n$, $\mathbf{d} \neq \mathbf{0}$, the set

$$\{\mathbf{v} \in \mathbb{R}^n : \mathbf{v}\cdot\mathbf{d} = 0\}$$

is a hyperplane: all vectors perpendicular to $\mathbf{d}$.



$\mathbb{R}^2$: a line



$\mathbb{R}^3$: a plane

1.3 Matrices and Their Column Spaces

Matrix with m rows, n columns: $m \times n$ matrix $(A, B, \ldots)$

3×2 matrix : $\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$

$m\times n$ matrix : $\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$

$A + B,\ cA$:

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + \begin{bmatrix}5 & 6\\7 & 8\end{bmatrix} = \begin{bmatrix}6 & 8\\10 & 12\end{bmatrix}$$

$$2\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}2 & 4\\6 & 8\end{bmatrix}$$

0 : zero matrix, $a_{ij} = 0$ for all i, j

Square matrix: $m = n$.

$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 5\end{bmatrix}$	$\begin{bmatrix}2 & 1 & -3\\0 & 4 & 7\\0 & 0 & 5\end{bmatrix}$	$\begin{bmatrix}2 & 0 & 0\\1 & 4 & 0\\-3 & 7 & 5\end{bmatrix}$	$\begin{bmatrix}2 & 1 & -3\\1 & 4 & 7\\-3 & 7 & 5\end{bmatrix}$
identity (symbol: I)	diagonal	upper triangular	lower triangular	symmetric
$a_{ii} = 1, a_{ij} = 0$ if $i \neq j$	$a_{ij} = 0$ if $i \neq j$	$a_{ij} = 0$ if $i > j$	$a_{ij} = 0$ if $i < j$	$a_{ij} = a_{ji}$

1.3.1 Matrix-vector multiplication

$$\underbrace{7\begin{bmatrix}1\\3\\5\end{bmatrix}+8\begin{bmatrix}2\\4\\6\end{bmatrix}}_{\text{combination}} = \begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix} = \underbrace{\begin{bmatrix}1\cdot7+2\cdot8\\3\cdot7+4\cdot8\\5\cdot7+6\cdot8\end{bmatrix}}_{\text{scalar products}}$$

$$A\mathbf{x} = \underbrace{\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\\vdots\\a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n\end{bmatrix}$$

$$\underbrace{x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots+x_n\mathbf{v}_n}_{\text{combination}} = \underbrace{\begin{bmatrix}|&|&&|\\\mathbf{v}_1&\mathbf{v}_2&\cdots&\mathbf{v}_n\\|&|&&|\end{bmatrix}}_{A\text{, column picture}}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = \underbrace{\begin{bmatrix}—&\mathbf{u}_1&—\\—&\mathbf{u}_2&—\\&\vdots&\\—&\mathbf{u}_m&—\end{bmatrix}}_{A\text{, row picture}}\underbrace{\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix}\mathbf{u}_1\cdot\mathbf{x}\\\mathbf{u}_2\cdot\mathbf{x}\\\vdots\\\mathbf{u}_m\cdot\mathbf{x}\end{bmatrix}}_{\text{scalar products}}$$

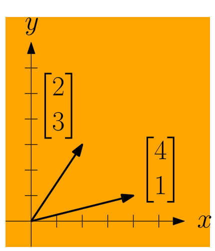
1.3.2 Column space: $\mathbf{C}(A)$

All combinations ("span") of the columns. If A is $m\times n$,

$$\mathbf{C}(A) = \{A\mathbf{x} : \mathbf{x}\in\mathbb{R}^n\} \subseteq \mathbb{R}^m. \qquad \text{Always: } \mathbf{0}\in\mathbf{C}(A).$$

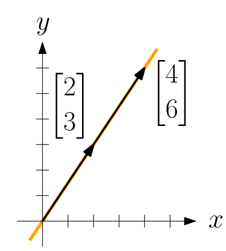
$$\mathbf{C}\left(\begin{bmatrix}2&4\\3&1\end{bmatrix}\right) = \mathbb{R}^2$$

(plane, 2-dim.)



$$\mathbf{C}\left(\begin{bmatrix}2&4\\3&6\end{bmatrix}\right) = \left\{c\begin{bmatrix}2\\3\end{bmatrix} : c\in\mathbb{R}\right\}$$

(a line, 1-dim.)

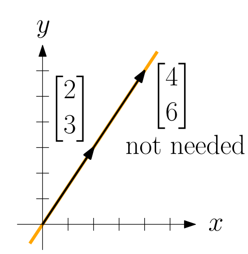


How many columns are needed to span $\mathbf{C}(A)$?

$$A = \begin{bmatrix}|&|&&|\\\mathbf{v}_1&\mathbf{v}_2&\cdots&\mathbf{v}_n\\|&|&&|\end{bmatrix}$$

Check $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$! If $\mathbf{v}_i$ is a combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$, then $\mathbf{v}_i$ is *dependent* (not needed): Every combination of $\mathbf{v}_1, \ldots, \mathbf{v}_i$ is already a combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$. *Proof:*

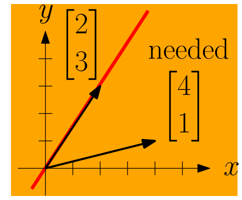
$$\begin{aligned}\underbrace{c_1\mathbf{v}_1+\cdots+c_i\mathbf{v}_i}_{\text{combination of } \mathbf{v}_1,\cdots,\mathbf{v}_i} &= c_1\mathbf{v}_1+\cdots+c_{i-1}\mathbf{v}_{i-1}+c_i\underbrace{(d_1\mathbf{v}_1+\cdots+d_{i-1}\mathbf{v}_{i-1})}_{\mathbf{v}_i}\\ &= \underbrace{(c_1+c_id_1)\mathbf{v}_1+\cdots+(c_{i-1}+c_id_{i-1})\mathbf{v}_{i-1}}_{\text{combination of } \mathbf{v}_1,\ldots,\mathbf{v}_{i-1}}\end{aligned}$$



Otherwise, $\mathbf{v}_i$ is *independent* (needed: "adds a dimension.")

Checking order doesn't matter: we always find the same number of independent columns (3.4).

For $\mathbf{v}_1$ ($i = 1$): $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$ contains no vectors. $\mathbf{0}$ is the only combination of no vectors. ("The sum of nothing is 0".)



1.3.3 (Linear) independence of vectors

Definition: Vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ are...

...(linearly) independent if...

(i) no vector is a combination of the previous ones. Or

(ii) no vector is a combination of the other ones. Or

(iii) there are no $c_1, c_2, \ldots, c_k$ besides $0, 0, \ldots, 0$ such that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k = \mathbf{0}.$$

...(linearly) dependent if...

(i′) some vector is a combination of the previous ones. Or

(ii′) some vector is a combination of the other ones. Or

(iii′) there are some $c_1, c_2, \ldots, c_k$ besides $0, 0, \ldots, 0$ such that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k = \mathbf{0}.$$

All say the same (are equivalent): (i) ⇔ (ii) ⇔ (iii). The opposites also: (i′) ⇔ (ii′) ⇔ (iii′).

Proof: (i′)⇒(ii′) (if (i′) is true, then (ii′) is true): clear ("previous ones" are "other ones"). (ii′)⇒(iii′): If

$$\mathbf{w}_i = c_1\mathbf{w}_1 + \cdots + c_{i-1}\mathbf{w}_{i-1} + c_{i+1}\mathbf{w}_{i+1} + \cdots + c_k\mathbf{w}_k, \quad \leftarrow \text{(ii′)}$$

then

$$c_1\mathbf{w}_1 + \cdots + c_{i-1}\mathbf{w}_{i-1} - 1\mathbf{w}_i + c_{i+1}\mathbf{w}_{i+1} + c_k\mathbf{w}_k = \mathbf{0}. \quad \leftarrow \text{(iii′)}$$

(iii′)⇒(i′): If there are some $c_1, c_2, \ldots, c_k$ besides $0, 0, \ldots, 0$ such that

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k = \mathbf{0} \quad \leftarrow \text{(iii′)}$$

take the *largest* i such that $c_i \neq 0$. Then $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_i\mathbf{w}_i = \mathbf{0}$ and hence

$$\mathbf{w}_i = -\frac{c_1}{c_i}\mathbf{w}_1 - \cdots - \frac{c_{i-1}}{c_i}\mathbf{w}_{i-1}. \quad \leftarrow \text{(i′)}$$

The columns of a matrix A are...

...independent if ...

(iii) there is no $\mathbf{x}$ besides $\mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$.

...dependent if ...

(iii′) there is some $\mathbf{x}$ besides $\mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$.

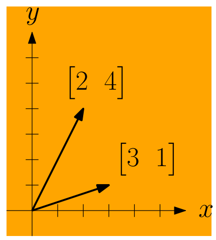
1.3.4 Rank: $\mathrm{rank}(A)$ = number of independent columns

$$\textbf{rank}\left(\begin{bmatrix}2 & 4\\3 & 1\end{bmatrix}\right)=2,\quad \textbf{rank}\left(\begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}\right)=1,\quad \textbf{rank}\left(\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right)=0.$$

Row space: $\mathbf{R}(A)$. All combinations of the rows

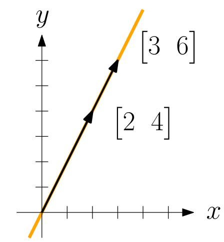
$$\mathbf{R}\left(\begin{bmatrix}2 & 4\\3 & 1\end{bmatrix}\right)=\mathbb{R}^2$$

(plane, 2-dim.)



$$\mathbf{R}\left(\begin{bmatrix}2 & 4\\3 & 6\end{bmatrix}\right)=\left\{c\begin{bmatrix}2 & 4\end{bmatrix}: c\in\mathbb{R}\right\}$$

(a line, 1-dim.)



In the examples, number of independent columns = number of independent rows. Coincidence? No (3.5)! Easy case: rank 1.

Matrices of rank 1. One independent column.

$$\begin{array}{c}\text{All columns}\\\text{of } A \text{ are}\\\text{multiples of}\end{array}\ \underbrace{\begin{bmatrix}v_1\\v_2\\\vdots\\v_m\end{bmatrix}}_{\neq\mathbf{0}}\ \begin{array}{c}\Rightarrow\\ \\ \Leftarrow\end{array}\ A=\underbrace{\begin{bmatrix}c_1v_1 & c_2v_1 & \cdots & c_nv_1\\c_1v_2 & c_2v_2 & \cdots & c_nv_2\\\vdots & \vdots & \ddots & \vdots\\c_1v_m & c_2v_m & \cdots & c_nv_m\end{bmatrix}}_{\text{rank 1: some } c_jv_i\neq 0}\ \begin{array}{c}\Rightarrow\\ \\ \Leftarrow\end{array}\ \begin{array}{c}\text{All rows}\\\text{of } A \text{ are}\\\text{multiples of}\end{array}\ \underbrace{\begin{bmatrix}c_1, c_2, \ldots, c_n\end{bmatrix}}_{\neq\mathbf{0}}$$

1.4 Matrix Multiplication AB and CR

$A: m\times k$ matrix; $B: k\times n$ matrix; $AB: m\times n$ matrix.

$$AB=\underbrace{\begin{bmatrix}\text{—} & \mathbf{u}_1 & \text{—}\\\text{—} & \mathbf{u}_2 & \text{—}\\ & \vdots & \\\text{—} & \mathbf{u}_m & \text{—}\end{bmatrix}}_{A,\text{ row picture}}\underbrace{\begin{bmatrix}| & | & & |\\\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\\| & | & & |\end{bmatrix}}_{B,\text{ column picture}}=\underbrace{\begin{bmatrix}\mathbf{u}_1\cdot\mathbf{v}_1 & \mathbf{u}_1\cdot\mathbf{v}_2 & \cdots & \mathbf{u}_1\cdot\mathbf{v}_n\\\mathbf{u}_2\cdot\mathbf{v}_1 & \mathbf{u}_2\cdot\mathbf{v}_2 & \cdots & \mathbf{u}_2\cdot\mathbf{v}_n\\\vdots & \vdots & \ddots & \vdots\\\mathbf{u}_m\cdot\mathbf{v}_1 & \mathbf{u}_m\cdot\mathbf{v}_2 & \cdots & \mathbf{u}_m\cdot\mathbf{v}_n\end{bmatrix}}_{mn\text{ scalar products}}$$

$$\begin{aligned}AB &= \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}1\cdot 0+2\cdot 1 & 1\cdot 1+2\cdot 0\\3\cdot 0+4\cdot 1 & 3\cdot 1+4\cdot 0\end{bmatrix}=\begin{bmatrix}2 & 1\\4 & 3\end{bmatrix} && \text{"column exchange"}\\BA &= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}=\begin{bmatrix}0\cdot 1+1\cdot 3 & 0\cdot 2+1\cdot 4\\1\cdot 1+0\cdot 3 & 1\cdot 2+0\cdot 4\end{bmatrix}=\begin{bmatrix}3 & 4\\1 & 2\end{bmatrix} && \text{"row exchange"}\end{aligned}$$

Square matrices: usually, $BA\neq AB$ (matrix multiplication is not commutative).
General matrices: BA can be undefined (if $m\neq n$), or of different size than AB.

Everything is matrix multiplication!

Vector-vector

Matrix-vector: $\underbrace{\begin{bmatrix}1&2\\3&4\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{2\times 1}=\underbrace{\begin{bmatrix}3\\7\end{bmatrix}}_{2\times 1}$ Scalar (inner) product: $\underbrace{\begin{bmatrix}1&2\end{bmatrix}}_{1\times 2}\underbrace{\begin{bmatrix}3\\4\end{bmatrix}}_{2\times 1}=\underbrace{\begin{bmatrix}11\end{bmatrix}}_{1\times 1}$

Vector-matrix: $\underbrace{\begin{bmatrix}1&1\end{bmatrix}}_{1\times 2}\underbrace{\begin{bmatrix}1&2\\3&4\end{bmatrix}}_{2\times 2}=\underbrace{\begin{bmatrix}4&6\end{bmatrix}}_{1\times 2}$ Outer product: $\underbrace{\begin{bmatrix}3\\4\end{bmatrix}}_{2\times 1}\underbrace{\begin{bmatrix}1&2\end{bmatrix}}_{1\times 2}=\underbrace{\begin{bmatrix}3&6\\4&8\end{bmatrix}}_{2\times 2}$ $\leftarrow$ rank 1

$$\underbrace{\begin{bmatrix}— & \mathbf{u}_1 B & —\\ — & \mathbf{u}_2 B & —\\ & \vdots & \\ — & \mathbf{u}_m B & —\end{bmatrix}}_{AB\text{, row picture}} = \underbrace{\begin{bmatrix}— & \mathbf{u}_1 & —\\ — & \mathbf{u}_2 & —\\ & \vdots & \\ — & \mathbf{u}_m & —\end{bmatrix}}_{A\text{, row picture}} \underbrace{\begin{bmatrix}| & | & & |\\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\\ | & | & & |\end{bmatrix}}_{B\text{, column picture}} = \underbrace{\begin{bmatrix}| & | & & |\\ A\mathbf{v}_1 & A\mathbf{v}_2 & \cdots & A\mathbf{v}_n\\ | & | & & |\end{bmatrix}}_{AB\text{, column picture}}$$

1.4.1 Distributivity and associativity

$$A(B+C)=AB+AC \text{ and } (B+C)D=BD+CD \qquad (AB)C=A(BC)=ABC.$$

More matrices: brackets don't matter: $(AB)(CD)=A((BC)D)=\cdots=ABCD$.

Distributivity: easy
Associativity: boring calculations with sums and products involving matrix entries
More matrices: needs proof!

1.4.2 $A = CR$

Finding the independent columns, revisited:

$$A=\begin{bmatrix}1&2&0&3\\2&4&1&4\\3&6&2&5\end{bmatrix}$$

	columns of A			
	$\begin{bmatrix}1\\2\\3\end{bmatrix}$	$\begin{bmatrix}2\\4\\6\end{bmatrix}$	$\begin{bmatrix}0\\1\\2\end{bmatrix}$	$\begin{bmatrix}3\\4\\5\end{bmatrix}$
	$\|\|$	$\|\|$	$\|\|$	$\|\|$
$\mathbf{v}_1$	$1\mathbf{v}_1$	$2\mathbf{v}_1$		$3\mathbf{v}_1$
$\mathbf{v}_2$				
$\mathbf{v}_3$			$1\mathbf{v}_3$	$-2\mathbf{v}_3$
$\mathbf{v}_4$				
independent?	yes	no	yes	no

$$A=\underbrace{\begin{bmatrix}1&0\\2&1\\3&2\end{bmatrix}}_{C}\underbrace{\begin{bmatrix}1&2&0&3\\0&0&1&-2\end{bmatrix}}_{R}$$

C: the independent columns

R: how to combine them to get all columns

Rank factorization: if A has r independent columns, then $\underbrace{A}_{m\times n}=\underbrace{C}_{m\times r}\underbrace{R}_{r\times n}$.

Efficient computation: (3.2)

R is unique: if $A=CR=CR'$, then $C(R-R')=0 \Rightarrow C\mathbf{w}=\mathbf{0}$ for every column $\mathbf{w}$ of $R-R' \Rightarrow \mathbf{w}=\mathbf{0}$, since the columns of C are independent (1.3.3).

Chapter 2

Solving Linear Equations $A\mathrm{x} = \mathrm{b}$

2.1 Elimination and back substitution

System of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}\right\} A\mathbf{x} = \mathbf{b} : \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A,\ m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}\in\mathbb{R}^n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{b}\in\mathbb{R}^m}$$

Given A and $\mathbf{b}$, find $\mathbf{x}$!

For now: $m = n$, A is square matrix.

2.1.1 Back substitution

If A upper triangular:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix}$$

	equation	substitution	solution
row 3	$7x_3 = 14$		$x_3 = 2$
row 2	$5x_2 + 6x_3 = 17$	$5x_2 + 12 = 17$	$x_2 = 1$
row 1	$2x_1 + 3x_2 + 4x_3 = 19$	$2x_1 + 11 = 19$	$x_1 = 4$

2.1.2 Elimination

General case: Transform $A\mathbf{x} = \mathbf{b}$ to $U\mathbf{x} = \mathbf{c}$ with same solution but upper triangular U (Gauss elimination). Then back substitution!

Row Operations

fat number: the **pivot** $\qquad A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 19 \\ 55 \\ 50 \end{bmatrix}$

subtract $2\cdot$(Row 1) from (Row 2): $\downarrow$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 0 & 5 & 6 \\ 2 & 8 & 17 \end{bmatrix} \qquad E_{21}\mathbf{b} = \begin{bmatrix} 19 \\ 17 \\ 50 \end{bmatrix}$$

subtract $1\cdot$(Row 1) from (Row 3): $\downarrow$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{5} & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \begin{bmatrix} 19 \\ 17 \\ 31 \end{bmatrix}$$

subtract $1\cdot$(Row 2) from (Row 3): $\downarrow$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}A}_{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \qquad \underbrace{E_{32}E_{31}E_{21}\mathbf{b}}_{\mathbf{c}} = \begin{bmatrix} 19 \\ 17 \\ 14 \end{bmatrix}$$

$\uparrow$ elimination matrices $\qquad$ done!

Less nice case:

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 8 & 17 \end{bmatrix} \qquad \mathbf{b} = \cdots$$

elimination in first column: $\downarrow$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{0} & 6 \\ 0 & 5 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \cdots$$

can't go on with pivot 0: exchange rows 2 and 3: $\downarrow$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \underbrace{P_{23}E_{31}E_{21}A}_{U} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & \mathbf{5} & 13 \\ 0 & 0 & 6 \end{bmatrix} \qquad \underbrace{P_{23}E_{31}E_{21}\mathbf{b}}_{\mathbf{c}} = \cdots$$

$\uparrow$ permutation matrix $\qquad$ done!

Ugly case:

$$A = \begin{bmatrix} \mathbf{2} & 3 & 4 \\ 4 & 6 & 14 \\ 2 & 3 & 17 \end{bmatrix} \qquad \mathbf{b} = \cdots$$

elimination in first column: $\downarrow$

$$E_{31}E_{21}A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 13 \end{bmatrix} \qquad E_{31}E_{21}\mathbf{b} = \cdots$$

no row exchange helps, give up for now!

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

also ugly!

Solving $U\mathbf{x} = \mathbf{c}$ also solves $A\mathbf{x} = \mathbf{b}$

Same solutions before and after each row operation!

$$
\begin{array}{ccccc}
\begin{array}{r}\mathbf{x} \text{ is a} \\ \text{solution} \\ \text{before:}\end{array} & A\mathbf{x} = \mathbf{b} & \begin{array}{c}\text{subtract } c\cdot(\text{Row } i) \text{ from (Row } j) \\ \text{or exchange rows } i \text{ and } j \\ \underbrace{\Longrightarrow}_{\text{do!}}\end{array} & \begin{array}{c}T\text{: matrix of the} \\ \text{row operation} \\ \underbrace{TA}_{A'}\mathbf{x} = \underbrace{T\mathbf{b}}_{\mathbf{b}'}\end{array} & \\
 & \Uparrow & & \Downarrow & \\
 & \begin{array}{c}\overbrace{T'A'}^{A}\mathbf{x} = \overbrace{T'\mathbf{b}'}^{\mathbf{b}} \\ T'\text{: matrix of the} \\ \text{row operation}\end{array} & \begin{array}{c}\overbrace{\Longleftarrow}^{\text{undo!}} \\ \text{add } c\cdot(\text{Row } i) \text{ to (Row } j) \\ \text{or exchange rows } i \text{ and } j\end{array} & A'\mathbf{x} = \mathbf{b}' : & \begin{array}{l}\mathbf{x} \text{ is a} \\ \text{solution} \\ \text{after}\end{array}
\end{array}
$$

Also holds if A is non-square.

Special case: $\mathbf{b} = \mathbf{0}$ ($\Rightarrow \mathbf{b}' = T\mathbf{b} = \mathbf{0}$): $\quad A\mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad A'\mathbf{x} = \mathbf{0}$

(In)dependence of columns is preserved

$$
\begin{array}{c}\text{The columns of} \\ A \text{ are dependent}\end{array} \overset{1.3.3}{\Longleftrightarrow} \begin{array}{c}\text{There is } \mathbf{x} \neq \mathbf{0} \\ \text{such that } A\mathbf{x} = \mathbf{0}\end{array} \Leftrightarrow \begin{array}{c}\text{There is } \mathbf{x} \neq \mathbf{0} \\ \text{such that } A'\mathbf{x} = \mathbf{0}\end{array} \overset{1.3.3}{\Longleftrightarrow} \begin{array}{c}\text{The columns of} \\ A' \text{ are dependent}\end{array}
$$

Ugly case in step j $\Rightarrow$ the first j columns are dependent

$$
j = 3 : \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\neq \mathbf{0}}^{\text{find}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

	equation	substitution	solution
			anything goes!
row 3	$0x_3 = 0$		$x_3 = 4$
row 2	$4x_2 + 5x_3 = 0$	$4x_2 + 20 = 0$	$x_2 = -5$
row 1	$x_1 + 2x_2 + 3x_3 = 0$	$x_1 + 2 = 0$	$x_1 = -2$

Also true in the original matrix A, because (in)dependence of columns is preserved.

2.1.3 Elimination succeeds $\Leftrightarrow$ the columns of A are independent

Elimination (allowing row exchanges) succeeds:

- $\Rightarrow$ U has nonzero diagonal elements (pivots).
- $\Rightarrow$ Every column of U is independent from the previous ones.
- $\Rightarrow$ The columns of U are independent (1.3.3).
- $\Rightarrow$ The columns of A are independent (2.1.2).

Elimination fails:

- $\Rightarrow$ The columns of some intermediate matrix are dependent (ugly case)
- $\Rightarrow$ The columns of A are dependent. (2.1.2).

2.2 Elimination Matrices and Inverse Matrices

Elimination:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix} \quad \underset{\text{undo!}}{\overset{\text{do!}}{\substack{\longrightarrow \\ \longleftarrow}}} \quad U = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\overbrace{E_{32}E_{31}E_{21}}^{E} A = U \qquad E_{ij} : \textit{do!} \qquad\qquad E_{ij}^{-1} : \textit{undo!} \qquad A = \overbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}^{E^{-1}} U$$

E_{21}: subtract $2\cdot$(Row 1) from (Row 2)
E_{31}: subtract $1\cdot$(Row 1) from (Row 3)
E_{32}: subtract $1\cdot$(Row 2) from (Row 3)

E_{21}^{-1}: add $2\cdot$(Row 1) to (Row 2)
E_{31}^{-1}: add $1\cdot$(Row 1) to (Row 3)
E_{32}^{-1}: add $1\cdot$(Row 2) to (Row 3)

An $n \times n$ matrix M is *invertible* if there is an $n \times n$ matrix M^{-1} (the inverse of M) such that

$$MM^{-1} = M^{-1}M = I \qquad \left(I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \right).$$

$$\begin{array}{rcl} M \cdot \ldots & : & \text{do something!} \\ M^{-1} \cdot \ldots & : & \text{undo it !} \\ I \cdot \ldots & : & \text{do nothing!} \end{array}$$

There can only be one inverse: If $MX = YM = I$, then $X = Y$, because

$$X = IX = (YM)X \underset{\substack{\uparrow \\ \text{associativity (1.4.1)}}}{=} Y(MX) = YI = Y.$$

Case 1×1: $M = \begin{bmatrix} x \end{bmatrix}$, $\quad M^{-1} = \begin{bmatrix} \frac{1}{x} \end{bmatrix}$ (if $x \neq 0$).
Case 2×2:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{if } ad - bc \neq 0)$$

2.2.1 The Inverse Theorem

Case $n \times n$:

$$\boxed{\begin{array}{cr} A \text{ is invertible} & \text{(i)} \\ \Leftrightarrow & \\ \text{For every } \mathbf{b} \in \mathbb{R}^n,\ A\mathbf{x} = \mathbf{b} \text{ has a unique solution } \mathbf{x} & \text{(ii)} \\ \Leftrightarrow & \\ \text{the columns of } A \text{ are independent} & \text{(iii)} \end{array}}$$

Proof:

(i) $\Rightarrow$ (ii): if A is invertible, then

- $A^{-1}\mathbf{b}$ solves $A\mathbf{x} = \mathbf{b}$: $\quad A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}.$
- Uniqueness: If $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x} = A^{-1}\mathbf{b}$: $\quad A^{-1}\mathbf{b} = A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = I\mathbf{x} = \mathbf{x}.$

(ii) $\Rightarrow$ (iii): if $A\mathbf{x} = \mathbf{0}$ has a unique solution ($\mathbf{0}$), the columns of A are independent (1.3.3).

(iii) $\Rightarrow$ (ii): If the columns of A are independent, elimination succeeds (2.1.3): $A\mathbf{x} = \mathbf{b} \Leftrightarrow U\mathbf{x} = \mathbf{c}$ (and U has nonzero diagonal elements). Back substitution: unique solution $\mathbf{x}$.

(ii) $\Rightarrow$ (i): If $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$, we find $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that

$$A\mathbf{v}_1 = \underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}_{\mathbf{e}_1}, A\mathbf{v}_2 = \underbrace{\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}}_{\mathbf{e}_2}, \ldots, A\mathbf{v}_n = \underbrace{\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}}_{\mathbf{e}_n} \quad \Rightarrow \quad A \underbrace{\begin{bmatrix}| & | & & |\\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\\ | & | & & |\end{bmatrix}}_{B} = \underbrace{\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}}_{I}.$$

So $AB = I$. Still need $BA = I$ to conclude that $B = A^{-1}$:

- $AI = IA = (AB)A = A(BA)$, hence $A(I - BA) = 0$ by distributivity (1.4.1).
- Columns of $I - BA$: $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$. Then $A\mathbf{w}_i = \mathbf{0}$ for all i.
- The columns of A are independent by (ii) $\Rightarrow$ (iii). Hence $\mathbf{w}_i = \mathbf{0}$ for all i. So $I - BA = 0$, meaning $BA = I$.

For *any* two $n \times n$ matrices A, B: If $AB = I$, then $BA = I$ (Exercise).

2.2.2 The inverse of a product AB

If A and B are $n \times n$ and invertible, then AB is also invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}. \quad \text{("undo" works in reverse order of "do")}$$

Proof: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$
Works for more matrices: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

2.3 Matrix Computations and $A = LU$

2.3.1 The cost of elimination

How many operations ($\cdot$, $/$, $+$, $-$) are needed to solve $A\mathbf{x} = \mathbf{b}$?

Elimination in step j. Subtract $\ell_{ij}\cdot$ (Row j) from (Row i):

$$\begin{array}{r|cccccc|ccc|c|} & & & \text{matrix} & & & & & \text{number of entries} & & \multicolumn{1}{l}{\text{right-hand side}} \\ & u_{11} & \cdots & & & & & & & & c_1 \\ & 0 & u_{22} & \cdots & & & & & & & c_2 \\ & 0 & 0 & \ddots & & & & & & & c_3 \\ \text{row } j & 0 & 0 & \cdots & \mathbf{u_{jj}} & \cdots & u_{jn} & \leftarrow & n-j+1 & 1 \;\; \rightarrow & c_j \\ \vdots & & & & & & & & & & \\ \text{row } i & 0 & 0 & \cdots & \star_{ij} & \cdots & \star_{in} & \leftarrow & n-j+1 & 1 \;\; \rightarrow & \star_i \end{array}$$

		for one i		for $i = j+1, \ldots, n$	
op.	where?	$A \to U$	$\mathbf{b} \to \mathbf{c}$	$A \to U$	$\mathbf{b} \to \mathbf{c}$
$/$	$\ell_{ij} = \star_{ij}/u_{jj}$	1		$(n-j)$	
$\cdot$	$\mathbf{r} = \ell_{ij}\cdot$ (Row j)	$(n-j+1)$	1	$(n-j)(n-j+1)$	$(n-j)$
$-$	(Row i) $-\mathbf{r}$	$(n-j+1)$	1	$(n-j)(n-j+1)$	$(n-j)$

Elimination in all steps $j = 1, \ldots, n-1$. Apply known formulas (sum of the first integers, sum of the first square numbers):

$A \to U$:

- Divisions: $\sum_{j=1}^{n-1}(n-j) = \frac{1}{2}(n^2-n)$
- Multiplications / Subtractions: $\sum_{j=1}^{n-1}(n-j)(n-j+1) = \frac{1}{3}(n^3-n)$

$\mathbf{b} \to \mathbf{c}$:

- Multiplications / Subtractions: $\sum_{j=1}^{n-1}(n-j) = \frac{1}{2}(n^2-n)$

Roughly $\mathbf{\frac{2}{3}n^3}$ operations for $A \to U$ and $\mathbf{n^2}$ for $\mathbf{b} \to \mathbf{c}$.

Back substitution. In row j of $U\mathbf{x} = \mathbf{c}$, substitute the already known values of $x_{j+1}, \ldots, x_n$ into

$$u_{jj}x_j + u_{j,j+1}x_{j+1} + \cdots + u_{jn}x_n = c_j$$

and solve for x_j:

$$x_j = \frac{1}{u_{jj}}\left(c_j - u_{j,j+1}x_{j+1} - \cdots - u_{jn}x_n\right).$$

op.	for one j	for $j = n, n-1, \ldots, 1$
$/$	1	n
$\cdot$	$(n-j)$	$\sum_{j=1}^{n}(n-j) = \frac{1}{2}(n^2-n)$
$-$	$(n-j)$	$\sum_{j=1}^{n}(n-j) = \frac{1}{2}(n^2-n)$

Roughly $\mathbf{n^2}$ operations.

$$\boxed{\text{Solving } A\mathbf{x} = \mathbf{b} \text{ (for one or more } \mathbf{b}'s\text{) takes roughly } \mathbf{\tfrac{2}{3}n^3} \text{ operations for } A \to U, \text{ and roughly } 2\mathbf{n^2} \text{ operations per } \mathbf{b} \text{ } (\mathbf{b} \to \mathbf{c}, \text{ back substitution}).}$$

2.3.2 The great factorization $A = LU$

Elimination: $A \to U$ (upper triangular). Assumption for now: no row exchanges!

Elimination in row i. Subtract $\ell_{ij}\cdot$ (Row j of U) from (Row i):

$$\begin{array}{r|cccccc|l}
 & u_{11} & \cdots & & & & & \leftarrow \text{finalized (in } U) \\
 & 0 & u_{22} & \cdots & & & & \leftarrow \text{finalized (in } U) \\
 & 0 & 0 & \ddots & & & & \qquad\vdots \\
\text{row } j & 0 & 0 & \cdots & \mathbf{u_{jj}} & \cdots & u_{jn} & \leftarrow \text{finalized (in } U) \\
\vdots & & & & & & & \\
\text{row } i & 0 & 0 & \cdots & \star_{ij} & \cdots & \star_{in} &
\end{array}$$

Happens in steps $j = 1, \dots, i-1$. How does (Row i) change in each step?

$$\begin{array}{cccll}
 & & & \text{(Row } i) \text{ of } A & \text{initially} \\
- & \ell_{i1} & \cdot & \text{(Row } 1) \text{ of } U & \text{step } 1 \\
- & \ell_{i2} & \cdot & \text{(Row } 2) \text{ of } U & \text{step } 2 \\
\vdots & & & & \\
- & \ell_{i,i-1} & \cdot & \text{(Row } i-1) \text{ of } U & \text{step } i-1 \\
= & & & \text{(Row } i) \text{ of } U & \text{in the end}
\end{array}$$

(Row i) of A is a combination of the first i rows of U. Matrix notation:

$$\text{(Row } i) \text{ of } A = \underbrace{\begin{bmatrix} \ell_{i1} & \ell_{i2} & \cdots & \ell_{i,i-1} & 1 & 0 & \cdots & 0 \end{bmatrix}}_{\text{row vector}} U.$$

$$A = \underbrace{\begin{bmatrix} 1 & & & \\ \ell_{21} & 1 & & \\ \vdots & & \ddots & \\ \ell_{n1} & \cdots & \ell_{n,n-1} & 1 \end{bmatrix}}_{L\text{, lower triangular}} \underbrace{\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix}}_{U\text{, upper triangular}}$$

In this notation, we omit 0's above/below the diagonal.

2.4 Permutations and Transposes

$A = LU$ fails if there are row exchanges. Is there a fix?

Fact: Reordering the rows of a matrix S reorders the rows of SA in the same way:

$$\underbrace{\begin{bmatrix} - & \mathbf{w}_1 & - \\ - & \mathbf{w}_2 & - \\ & \vdots & \\ - & \mathbf{w}_m & - \end{bmatrix}}_{S} A = \underbrace{\begin{bmatrix} - & \mathbf{w}_1 A & - \\ - & \mathbf{w}_2 A & - \\ & \vdots & \\ - & \mathbf{w}_m A & - \end{bmatrix}}_{SA}$$

Example: exchange rows $1, 2$ of $S \to S'$

$$\underbrace{\begin{bmatrix} - & \mathbf{w}_2 & - \\ - & \mathbf{w}_1 & - \\ & \vdots & \\ - & \mathbf{w}_m & - \end{bmatrix}}_{S'} A = \underbrace{\begin{bmatrix} - & \mathbf{w}_2 A & - \\ - & \mathbf{w}_1 A & - \\ & \vdots & \\ - & \mathbf{w}_m A & - \end{bmatrix}}_{S'A}$$

Permutation matrix P: reordering (*permutation*) of the rows of I.
PA: permutation of the rows of $IA = A$.
$P\mathbf{x}$: permutation of the the entries of $\mathbf{x}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

If P, P' are permutation matrices, then also PP': reordering twice is another reordering.

There are $n! = 1 \cdot 2 \cdots n$ permutation matrices ($n \times n$), since n things can be ordered in $n!$ ways:

n	$n!$	orderings
1	1	1
2	2	$12, 21$
3	6	$123, 132, 213, 231, 312, 321$
4	24	$1234, 1243, \ldots$

2.4.1 The $PA = LU$ factorization

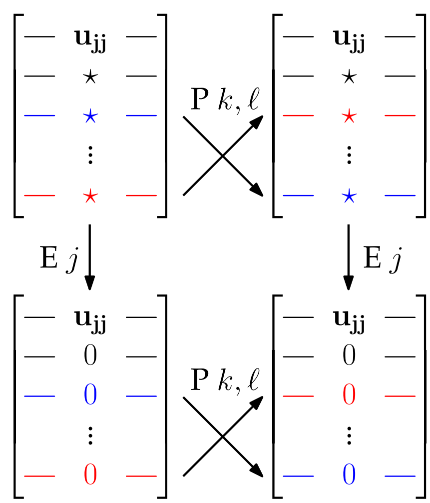
Idea: move all row exchanges to the beginning ($A \to PA$), then we can eliminate without row exchanges ($PA = LU$).

Notation:

E j	:	do all elimination steps in column j
P k, ℓ	:	exchange rows k and ℓ

Example ($\uparrow$: move row exchange up!):

$A \to U$			$PA = LU$	
E 1	P 2,5	P 2,5	P 2,5	exchange rows 2 and 5
P 2,5$\uparrow$	E 1	E 1	P 3,4	and then rows 3 and 4
E 2	E 2	P 3,4$\uparrow$	E 1	
P 3,4	P 3,4$\uparrow$	E 2	E 2	
1	2	3	$\leftarrow$ move	



Why it works:

E j
P k, ℓ has the same effect as P k, ℓ
E j if $k, \ell > j$.

2.4.2 The transpose of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & \underset{\substack{\uparrow \\ A_{23}}}{6} \end{bmatrix} \qquad \leftarrow \text{reflection along } {''}\diagdown{''} \rightarrow \qquad A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & \underset{\substack{\uparrow \\ (A^\top)_{32}}}{6} \end{bmatrix}$$

$$\begin{array}{rcl} \text{row } i \text{ of } A & = & \text{column } i \text{ of } A^\top \\ \text{column } j \text{ of } A & = & \text{row } j \text{ of } A^\top \end{array} \qquad \begin{array}{rcl} A_{ij} & = & (A^\top)_{ji} \\ (A^\top)^\top & = & A \end{array} \qquad \begin{array}{l} \text{Scalar product:} \\ \mathbf{v} \cdot \mathbf{w} = \underbrace{\mathbf{v}^\top}_{1\times n} \underbrace{\mathbf{w}}_{n\times 1} \end{array}$$

Transpose of a product: $(AB)^\top = B^\top A^\top$

$$AB \quad \leftarrow \text{reflection along } {''}\diagdown{''} \rightarrow \quad B^\top A^\top :$$

$$\underbrace{\overset{(AB)_{ij}}{\left(\underbrace{\begin{bmatrix} — & \mathbf{u}_1 & — \\ — & \mathbf{u}_2 & — \\ & \vdots & \\ — & \mathbf{u}_m & — \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}}_{B} \right)_{ij}}}_{\mathbf{u}_i \cdot \mathbf{v}_j} = \underbrace{\overset{(B^\top A^\top)_{ji}}{\left(\underbrace{\begin{bmatrix} — & \mathbf{v}_1 & — \\ — & \mathbf{v}_2 & — \\ & \vdots & \\ — & \mathbf{v}_n & — \end{bmatrix}}_{B^\top} \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \\ | & | & & | \end{bmatrix}}_{A^\top} \right)_{ji}}}_{\mathbf{v}_j \cdot \mathbf{u}_i}$$

Works for more matrices: $(ABC)^\top = C^\top B^\top A^\top$.

Transpose of the inverse: $(A^{-1})^\top = (A^\top)^{-1}$

$$\begin{array}{ccccccc} & & AA^{-1} & = & I & & \\ & & & \Downarrow & & & \\ (A^{-1})^\top A^\top & = & (AA^{-1})^\top & = & I^\top & = & I \\ & & & \Downarrow & & & \\ & & \multicolumn{5}{l}{(A^{-1})^\top \text{ is the inverse of } A^\top} \end{array}$$

Permutation matrix: $P^{-1} = P^\top$. Rows of P: $\mathbf{p}_1, \ldots, \mathbf{p}_n$ (reordering of rows of I).
Each $\mathbf{p}_i$ has a single 1 at a different position $\Rightarrow$ $\mathbf{p}_i \cdot \mathbf{p}_i = 1$, $\mathbf{p}_i \cdot \mathbf{p}_j = 0$ for $i \neq j$.

$$\underbrace{\left(\underbrace{\begin{bmatrix} — & \mathbf{p}_1 & — \\ — & \mathbf{p}_2 & — \\ & \vdots & \\ — & \mathbf{p}_n & — \end{bmatrix}}_{P\text{, row picture}} \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \\ | & | & & | \end{bmatrix}}_{P^T\text{, column picture}} \right)_{ij}}_{\mathbf{p}_i \cdot \mathbf{p}_j} = I_{ij} \quad \Leftrightarrow \quad PP^\top = I.$$

2.4.3 Symmetric matrices

S is symmetric if $S = S^\top$ (such S must be square).

$$S = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 4 & 7 \\ -3 & 7 & 5 \end{bmatrix}$$

If S is symmetric, then also S^{-1} (if it exists):

$$(S^{-1})^\top = (S^\top)^{-1} = S^{-1}.$$

For *every* matrix A, both $A^\top A$ and $AA^\top$ are symmetric:

$$(A^\top A)^\top = A^\top (A^\top)^\top = A^\top A, \qquad (AA^\top)^\top = (A^\top)^\top A^\top = AA^\top.$$

2.4.4 Symmetric LU-factorization

Normal elimination step:
subtract 2·(Row 1)
from (Row 2)

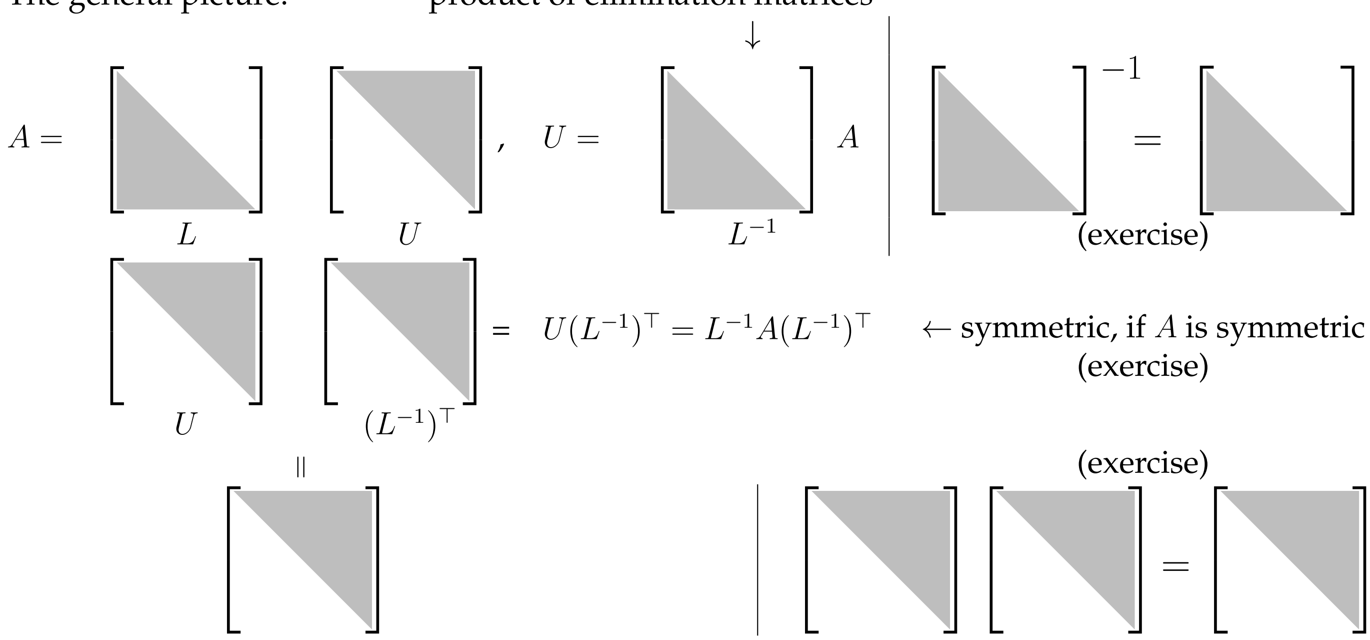
$$\underbrace{E_{21}}_{L^{-1}} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}}_{A,\text{ symmetric}} = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{U}$$

Now add this extra step:
subtract 2·(<u>Column</u> 1)
from (<u>Column</u> 2)

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}_{U} \underbrace{E_{21}^\top}_{(L^{-1})^\top} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{D,\text{ diagonal}}$$

$$\begin{array}{cc} U = L^{-1}A & A = LU \\ \Downarrow & \Downarrow \\ D = L^{-1}A(L^\top)^{-1} & \boxed{A = LDL^\top} \\ \Uparrow & \Uparrow \\ D = U(L^{-1})^\top & U = DL^\top \\ \quad = U(L^\top)^{-1} & \end{array}$$

The general picture:



$D = U(L^{-1})^\top$ is upper triangular and symmetric $\Rightarrow$ D is diagonal.

$$D = L^{-1}A(L^{-1})^\top \quad \rightarrow \quad \boxed{A = LDL^\top}$$

Chapter 3

The Four Fundamental Subspaces

3.1 Vector Spaces and Subspaces

3.1.1 Examples of vector spaces

There is more than $\mathbb{R}^2, \mathbb{R}^3, \ldots$

Vector space: (abstract) concept of things that we can do with vectors

$\mathbb{R}^2, \mathbb{R}^3, \ldots$: examples.

concept	number type	vector space
things that we can do with…	…numbers: calculations! $a+b,\ a-b,\ a\cdot b,\ a/b$	…vectors: combinations! $\mathbf{v}+\mathbf{w}$, $c\cdot\mathbf{v}$
examples	$\mathbb{N}$ (natural numbers)	$\mathbb{R}^2$
	$\mathbb{Z}$ (integers)	$\mathbb{R}^3$
	$\mathbb{Q}$ (rational numbers)	$\mathbb{C}^3$ (complex vectors)
	$\mathbb{R}$ (real numbers)	$\mathbb{R}^{2\times 2}$ (2×2 matrices; $A+B, cA$ (1.3))
	$\mathbb{C}$ (complex numbers)	$\mathbb{R}^{\mathbb{R}}$ (functions $f:\mathbb{R}\to\mathbb{R}$)
	$\{0,1\}$ (bits)	$\{0,1\}^n$ (bit vectors)
	$\vdots$	$\vdots$

We mostly (but not only) care about $\mathbb{R}^2, \mathbb{R}^3, \ldots$ and their subspaces.

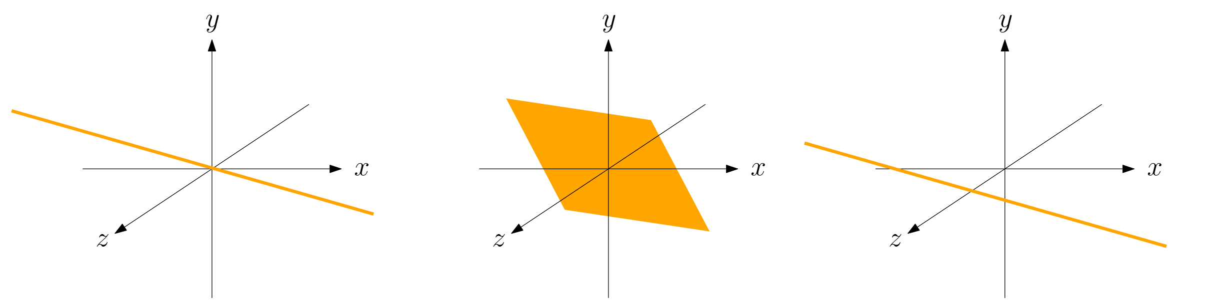
3.1.2 Subspaces of vector spaces

V: vector space. **Subspace**: nonempty $U \subseteq V$ satisfying: if $\mathbf{v}, \mathbf{w} \in U$ and c is a scalar, then

$$\text{(i) } \mathbf{v}+\mathbf{w} \in U \qquad \text{(ii) } c\mathbf{v} \in U.$$

Every subspace U contains $\mathbf{0}$: take some $\mathbf{u} \in U$, then $0\mathbf{u} = \mathbf{0} \in U$ by (ii).

Smallest subspace: $U = \{\mathbf{0}\}$. Largest subspace: $U = V$.



subspaces: line through $\mathbf{0}$ plane through $\mathbf{0}$ not a subspace: misses $\mathbf{0}$

A subspace of a vector space is itself a vector space.

Two subspaces of $V = \mathbb{R}^{2\times 2}$:

U_1: all symmetric matrices $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ U_2: all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$

3.1.3 The column space of A

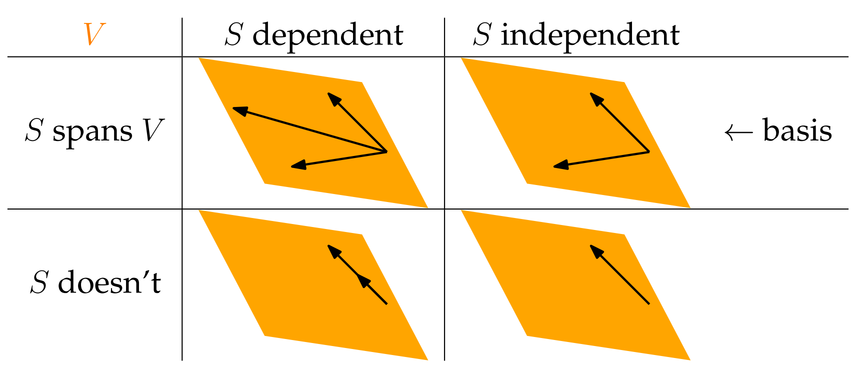
$$\mathbf{C}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$$

is a subspace of $\mathbb{R}^m$: If $\mathbf{v}, \mathbf{w} \in \mathbf{C}(A)$ and c a scalar, then $A\mathbf{x} = \mathbf{v}$ and $A\mathbf{y} = \mathbf{w}$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Hence,

$$\text{(i) } \mathbf{v} + \mathbf{w} = A(\underbrace{\mathbf{x} + \mathbf{y}}_{\in \mathbb{R}^n}) \in \mathbf{C}(A) \qquad \text{(ii) } c\mathbf{v} = A(\underbrace{c\mathbf{x}}_{\in \mathbb{R}^n}) \in \mathbf{C}(A)$$

3.1.4 The columns of A span the vector space $C(A)$

Span, Basis	Example
V: vector space	$\mathbf{C}(A)$
S: sequence of vectors in V	the columns of A
S **spans** V: V = all combinations of S	the columns span $\mathbf{C}(A)$
S **basis** of V: S independent, S spans V	the independent columns: basis of $\mathbf{C}(A)$



3.2 Computing the Nullspace by Elimination: $A = CR$

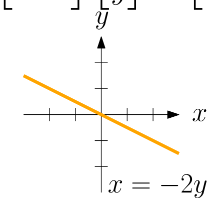
Nullspace of $(m \times n)$ matrix A: all solutions of $A\mathbf{x} = \mathbf{0}$

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \quad \text{(subspace of } \mathbb{R}^n\text{)}$$

If all columns are independent: $\mathbf{N}(A) = \{\mathbf{0}\}$
"Computing" a subspace: find a basis of it!
For $\mathbf{N}(A)$, we do this by computing $A = CR$ (1.4.2):

$$\mathbf{N}\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) : \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{gathered} x + 2y = 0 \\ 3x + 6y = 0 \\ \Updownarrow \\ x = -2y \end{gathered}$$



$$A = \begin{bmatrix} | & | & | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 & \mathbf{v}_7 \\ | & | & | & | & | & | & | \end{bmatrix} \quad \rightarrow \quad C = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \\ | & | & | \end{bmatrix} \text{ (the independent columns)}$$

$$\downarrow \qquad \mathbf{v}_1 = 1\mathbf{v}_1 \qquad \mathbf{v}_4 = r_{14}\mathbf{v}_1 + r_{24}\mathbf{v}_3$$

$$R = \begin{bmatrix} 1 & r_{12} & 0 & r_{14} & r_{15} & 0 & r_{17} \\ & & 1 & r_{24} & r_{25} & 0 & r_{27} \\ & & & & & 1 & r_{37} \end{bmatrix} \quad \text{(how to combine them to get all columns)}$$

R is in **reduced row echelon form**:

$$\begin{array}{|cccccc} 1 & 0 & 0 & 0 & 0 & \\ & 1 & 0 & 0 & 0 & \\ & & 1 & 0 & 0 & \\ 0 & & & 1 & 0 & \\ & & & & 1 & \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 & \cdots \end{array}$$

(standard unit vectors)

Plan:

Transform A to R using (Gauss-Jordan) elimination; we get C on the way.

Row operations don't change solutions (2.1.2): $A\mathbf{x} = \mathbf{0} \Leftrightarrow R\mathbf{x} = \mathbf{0}$, $\mathbf{N}(A) = \mathbf{N}(R)$.

Read a basis of $\mathbf{N}(R)$ off R.

The basis of $\mathbf{N}(R)$

Example: "free variables"

$$R = \underbrace{\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}}_{(1.4.2)} \quad \Bigg| \quad R\mathbf{x} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}}_{F} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -3 \\ 0 & 2 \end{bmatrix}}_{-F} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

Two special solutions: set the free variables $\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbf{x}$	every solution...	...is a combination of the two special independent solutions. Since they span $\mathbf{N}(R)$, they are a basis.
$\begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_3 \end{bmatrix}$	$\begin{bmatrix} x_2 \\ x_4 \\ -F \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \end{bmatrix}$	$= x_2 \left(\begin{bmatrix} 1 \\ 0 \\ -F \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} \right) + x_4 \left(\begin{bmatrix} 0 \\ 1 \\ -F \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \right)$

$$\Downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

General case: R is $(r \times n)$.

$\mathbf{x}_I$: vector of the r variables for $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r$
$\mathbf{x}_F$: vector of the $n-r$ other variables (free variables)

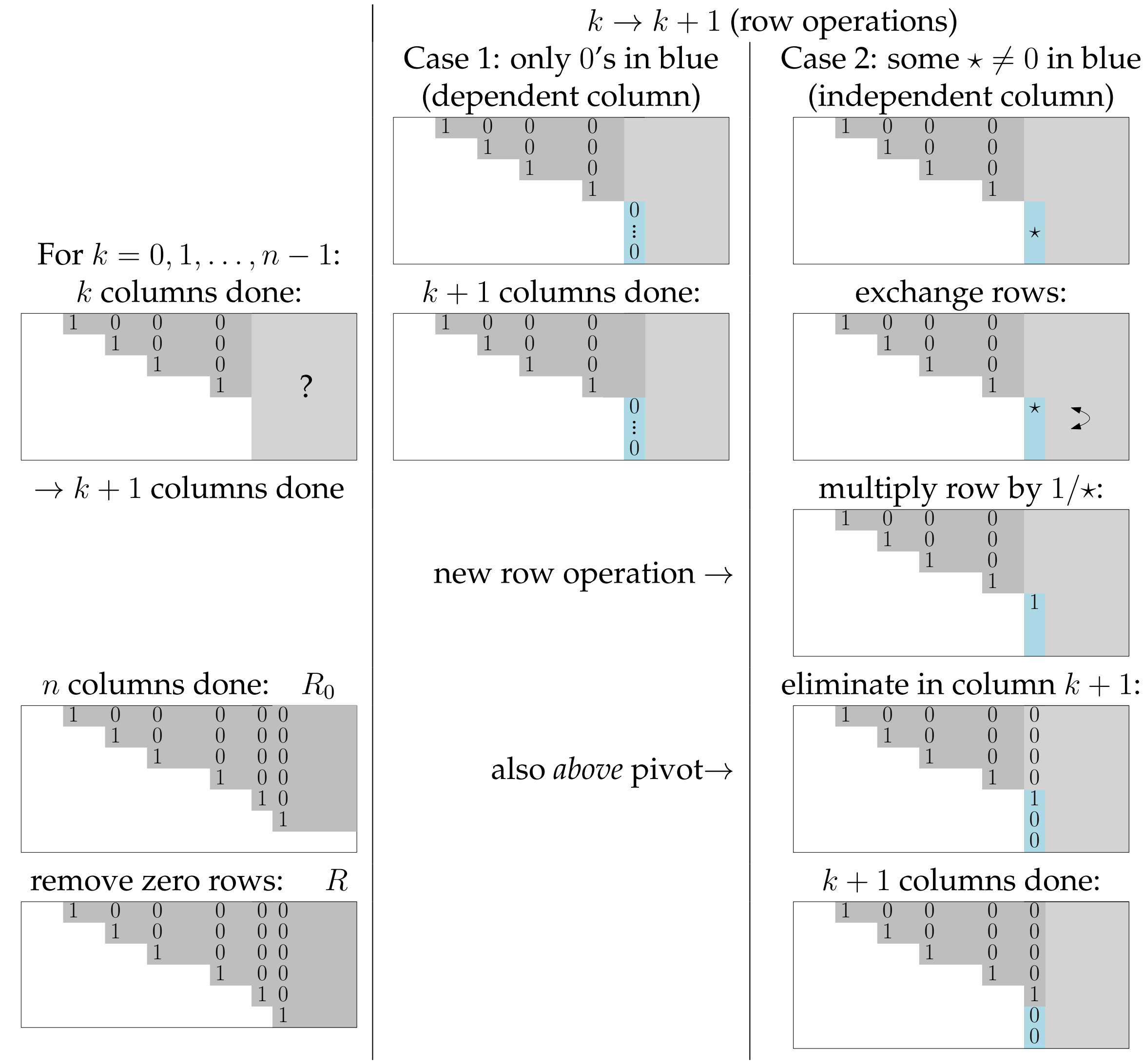
general	example
$r \times n$	2×4
$\mathbf{x}_F$	$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$
$\mathbf{x}_I$	$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$

$$R\mathbf{x} = I\mathbf{x}_I + F\mathbf{x}_F = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{x}_I = -F\mathbf{x}_F$$

$n-r$ special solutions: set the free variables $\mathbf{x}_F$ to $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-r}$

$\mathbf{x}$	every solution...	...is a combination of the $n-r$ special independent solutions. Since they span $\mathbf{N}(R)$, they are a basis.
$\begin{matrix} \mathbf{x}_F \\ \mathbf{x}_I \end{matrix}$	$\begin{matrix} \mathbf{x}_F \\ -F\mathbf{x}_F \end{matrix}$	$= (\mathbf{x}_F)_1 \begin{pmatrix} \mathbf{e}_1 \\ -F\mathbf{e}_1 \end{pmatrix} + (\mathbf{x}_F)_2 \begin{pmatrix} \mathbf{e}_2 \\ -F\mathbf{e}_2 \end{pmatrix} + \cdots + (\mathbf{x}_F)_{n-r} \begin{pmatrix} \mathbf{e}_{n-r} \\ -F\mathbf{e}_{n-r} \end{pmatrix}$

3.2.1 Elimination column by column: the steps from A to R_0



3.2.2 The matrix factorization $A = CR$ and the nullspace

$A \to R_0 \to R$ gives the same R as in $A = CR$ (1.4.2):

$$A = CR \qquad\qquad\qquad \text{elimination } A \to R_0$$

$$A = \begin{bmatrix} | & | & | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 & \mathbf{v}_7 \\ | & | & | & | & | & | & | \end{bmatrix} \quad \to \quad R_0 = \begin{bmatrix} | & | & | & | & | & | & | \\ \mathbf{e}_1 & \mathbf{w}_2 & \mathbf{e}_2 & \mathbf{w}_4 & \mathbf{w}_5 & \mathbf{e}_3 & \mathbf{w}_7 \\ | & | & | & | & | & | & | \end{bmatrix}$$

$$\downarrow \qquad \mathbf{v}_1 = 1\mathbf{v}_1 \qquad \boxed{\mathbf{v}_4 = r_{14}\mathbf{v}_1 + r_{24}\mathbf{v}_3} \quad \overset{(\star)}{\Longleftrightarrow} \quad \boxed{\mathbf{w}_4 = r_{14}\mathbf{e}_1 + r_{24}\mathbf{e}_2}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$R = \begin{bmatrix} 1 & r_{12} & 0 & r_{14} & r_{15} & 0 & r_{17} \\ & & 1 & r_{24} & r_{25} & 0 & r_{27} \\ & & & & & 1 & r_{37} \end{bmatrix} \quad \leftarrow \quad R_0 = \begin{bmatrix} 1 & r_{12} & 0 & r_{14} & r_{15} & 0 & r_{17} \\ & & 1 & r_{24} & r_{25} & 0 & r_{27} \\ & & & & & 1 & r_{37} \\ & & & & & & \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} r_{14} \\ 0 \\ r_{24} \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} : \quad \boxed{\mathbf{v}_4 = r_{14}\mathbf{v}_1 + r_{24}\mathbf{v}_3} \Leftrightarrow A\mathbf{x} = \mathbf{0} \overset{(2.1.2)}{\Longleftrightarrow} R_0\mathbf{x} = \mathbf{0} \Leftrightarrow \boxed{\mathbf{w}_4 = r_{14}\mathbf{e}_1 + r_{24}\mathbf{e}_2} \quad (\star)$$

3.3 The Complete Solution to $A\mathbf{x} = \mathbf{b}$

As in (2.1.2), apply row operations also to $\mathbf{b}$ ($A \to R_0, \mathbf{b} \to \mathbf{c}$). Solutions don't change:

$$A\mathbf{x} = \mathbf{b} \;\Leftrightarrow\; R_0\mathbf{x} = \mathbf{c} \;\Leftrightarrow\; \left.\begin{matrix} R\mathbf{x} & = & \mathbf{d} \\ 0 & = & \star \\ \vdots & \vdots & \vdots \\ 0 & = & \star \end{matrix}\right\} \mathbf{c}$$

$$A\mathbf{x} = \mathbf{0} \;\Leftrightarrow\; R\mathbf{x} = \mathbf{0}$$

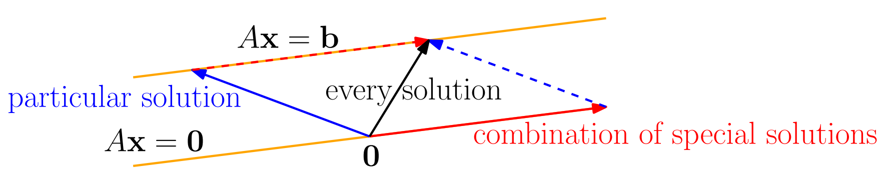
If some $\star \neq 0$, no solution!

Otherwise, solve

$$R\mathbf{x} = I\mathbf{x}_I + F\mathbf{x}_F = \mathbf{d} \quad\Leftrightarrow\quad \mathbf{x}_I = \mathbf{d} - F\mathbf{x}_F$$

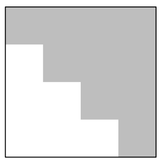
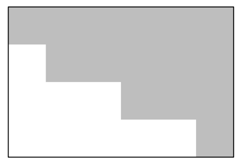
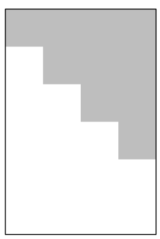
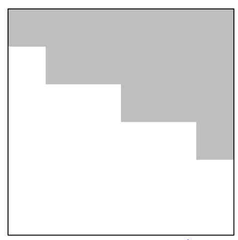
Particular solution: set the free variables $\mathbf{x}_F$ to $\mathbf{0}$

$\mathbf{x}$	every solution…	…is a particular solution of $R\mathbf{x} = \mathbf{d}$, plus a combination of the $n-r$ special solutions of $R\mathbf{x} = \mathbf{0}$
$\begin{matrix}\mathbf{x}_F \\ \mathbf{x}_I\end{matrix}$	$\begin{matrix}\mathbf{x}_F \\ \mathbf{d} - F\mathbf{x}_F\end{matrix}$	$= \begin{matrix}\mathbf{0} \\ \mathbf{d}\end{matrix} + (\mathbf{x}_F)_1 \begin{pmatrix} \mathbf{e}_1 \\ -F\mathbf{e}_1 \end{pmatrix} + (\mathbf{x}_F)_2 \begin{pmatrix} \mathbf{e}_2 \\ -F\mathbf{e}_2 \end{pmatrix} + \cdots + (\mathbf{x}_F)_{n-r} \begin{pmatrix} \mathbf{e}_{n-r} \\ -F\mathbf{e}_{n-r} \end{pmatrix}$



3.3.1 Number of solutions of $A\mathbf{x} = \mathbf{b}$

$$\begin{array}{lllll|ll}
\text{rank (number of independent columns)} & & & & = r & r \le n & \\
\downarrow & & & & & \Downarrow & \\
A\mathbf{x} = \mathbf{b} & \rightarrow & R_0\mathbf{x} = \mathbf{c} & \rightarrow & R\mathbf{x} = \mathbf{d} & r \le \min(m,n) & \text{full rank: } r = \min(m,n) \\
\uparrow & & \uparrow & & \uparrow & \Uparrow & \\
(m \times n) & & (m \times n) & & (r \times n) & r \le m &
\end{array}$$

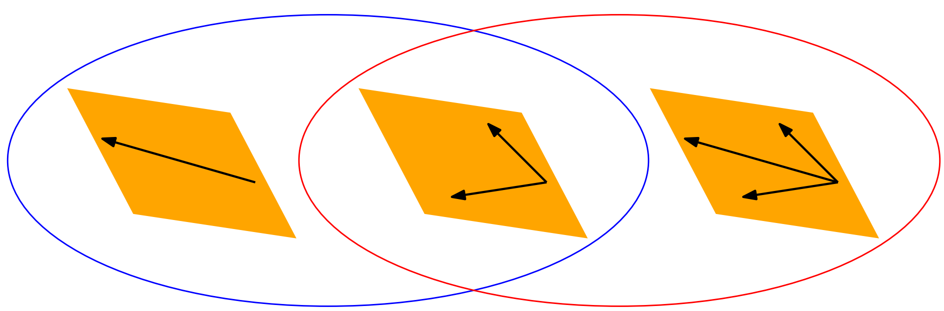
R_0	$r = n$ (full rank)	$r < n$ (dependent columns)	
$r = m$ (full rank)	invertible  1 solution	underdetermined  ∞ many solutions	← free variables
$r < m$ (zero rows)	overdetermined  0 or 1 solution	 0 or ∞ many solutions	← free variables

depending on $\mathbf{c}$ (if some $\star \neq 0$, then 0)

3.4 Independence, Basis, and Dimension

V: vector space

S: sequence of vectors in V



independent basis of V spans V

$\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ (columns of I): standard basis of $\mathbb{R}^n$.

The columns of *any* invertible $n \times n$ matrix A are a basis of $\mathbb{R}^n$. They are independent and spanning: for every $\mathbf{b} \in \mathbb{R}^n$, $A\mathbf{x} = \mathbf{b}$ has a solution (2.2.1).

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a basis of V, then every $\mathbf{v} \in V$ is a *unique* combination.

Proof: if $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$, then $\mathbf{0} = (a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n$. By independence, $a_1 - b_1 = \cdots = a_n - b_n = 0$.

Every basis of V has the same number of vectors. This number is the dimension $\dim(V)$ of V.

Proof (by contradiction):
Suppose there is a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ and a larger basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$. A basis spans $V \Rightarrow$ each $\mathbf{w}_j$ is a combination of the $\mathbf{v}_i$'s:

$$\mathbf{w}_j = \underbrace{\star\,\mathbf{v}_1 + \star\,\mathbf{v}_2 + \cdots + \star\,\mathbf{v}_m}_{\text{vector } \mathbf{x}_j \text{ with } m \text{ numbers}}$$

Matrix notation:

$$\underbrace{\begin{bmatrix}\mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n\end{bmatrix}}_{B} = \underbrace{\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m\end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{bmatrix}}_{X,\ m\times n}$$

$\mathbf{rank}(X) \le \min(m, n) = m < n$, so the columns of X are dependent (3.3.1): there is $\mathbf{c} \neq \mathbf{0}$ such that $X\mathbf{c} = \mathbf{0}$. Then $B\mathbf{c} = AX\mathbf{c} = A\mathbf{0} = \mathbf{0} \;\Leftrightarrow\; c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n = 0$, so the $\mathbf{w}_j$'s are dependent and *not* a basis. Contradiction!

Works for all vector spaces, not only (subspaces of) $\mathbb{R}^n$: consider A, B as $1 \times m, 1 \times n$ with vector entries (column vectors, or other objects).

3.4.1 Bases (for Matrix Spaces)

vector space	basis	dimension
$\mathbb{R}^n$	$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$	n
all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	4
diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	2
symmetric matrices $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	3
$\{0\}$	$\emptyset$ (empty set)	0

There are no independent vectors in $\{0\}$, so the basis must be empty. 0 is a combination of $\emptyset$ (sum of nothing = 0).

3.5 Dimensions of the Four Subspaces

A: $m \times n$ matrix (m rows, n columns).

This section:

subspace	of	definition	dimension
$\mathbf{C}(A)$	$\mathbb{R}^m$	combinations of the columns of A	$r = \mathbf{rank}(A)$
$\mathbf{R}(A) = \mathbf{C}(A^\top)$	$\mathbb{R}^n$	combinations of the rows of A = columns of $A^\top$	r
$\mathbf{N}(A)$	$\mathbb{R}^n$	solutions of $A\mathbf{x} = \mathbf{0}$	$n - r$
$\mathbf{N}(A^\top)$	$\mathbb{R}^m$	solutions of $A^\top\mathbf{y} = \mathbf{0}$	$m - r$

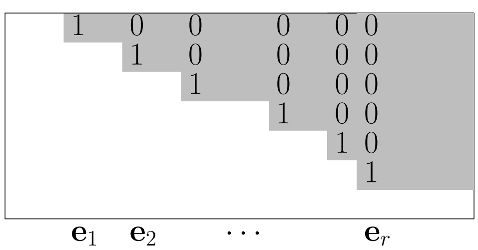
Row space $\mathbf{R}(A) = \mathbf{C}(A^\top)$

Gauss-Jordan: $A \to R_0$ by row operations:

- subtract $c\cdot$(Row i) from (Row j)
- exchange (Row i) and (Row j)
- multiply (Row i) with $c \neq 0$

Exercise: Row operations don't change the row space! $\mathbf{R}(A) = \mathbf{R}(R_0)$.

R_0:



r independent rows that span the row space: basis of $\mathbf{R}(R_0)$

$\dim(\mathbf{R}(A)) = \dim(\mathbf{R}(R_0)) = r$

zero rows

For every matrix: Number of independent rows = number of independent columns!
We knew this for rank-1 matrices ($r = 1$): (1.3.4)

Nullspace $\mathbf{N}(A)$

Gauss-Jordan: $A \to R_0 \to R$ (remove zero rows of R_0).
Row operations don't change the nullspace (2.1.2):
$A\mathbf{x} = \mathbf{0} \Leftrightarrow R_0\mathbf{x} = \mathbf{0} \Leftrightarrow R\mathbf{x} = \mathbf{0}$ $\mathbf{N}(A) = \mathbf{N}(R)$.
Already found a basis of $\mathbf{N}(R)$ with $n - r$ vectors (3.2). $\dim(N(A)) = n - r$.

Left nullspace $\mathbf{N}(A^\top)$

As previously shown for every matrix: $\dim$(nullspace) $=$ number of columns $-$ rank.
Apply this to $A^\top$: $\dim \mathbf{N}(A^\top) = m - \dim(C(A^\top)) = m - r$.
Why "left"? : all solutions of $A^\top\mathbf{y} = \mathbf{0}$ = all solutions of $\mathbf{y}^\top A = \mathbf{0}^\top$.

Chapter 4

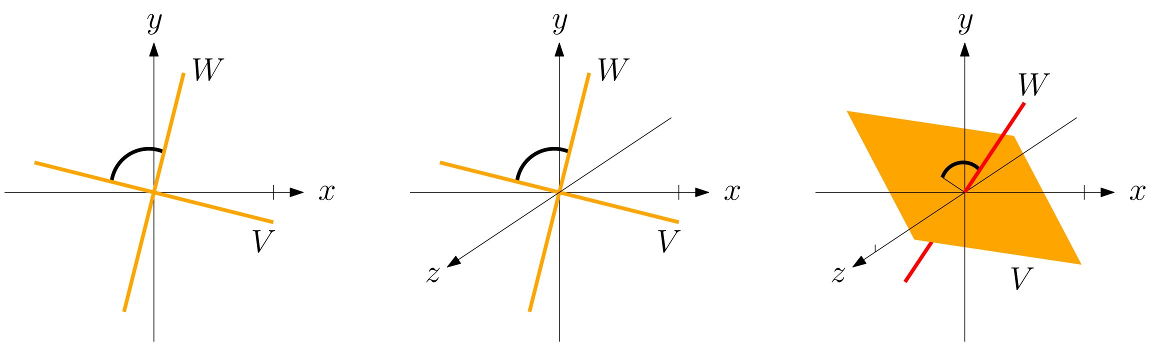
Orthogonality

4.1 Orthogonality of vectors and subspaces

Recall (1.2.3, 2.4.2): $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are perpendicular or orthogonal if $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^\top \mathbf{w} = 0$.

$$\underbrace{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\mathbf{v}\cdot\mathbf{w}} = \underbrace{\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\mathbf{v}^\top\mathbf{w}} = 0.$$

$\boxed{\text{Two subspaces } V \text{ and } W \text{ of } \mathbb{R}^n \text{ are orthogonal if } \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{v} \in V \text{ and all } \mathbf{w} \in W.}$



If A is $m \times n$:

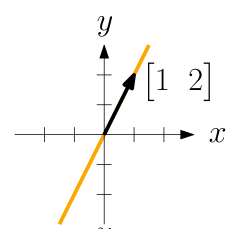
- $\mathbf{N}(A)$ and $\mathbf{R}(A) = \mathbf{C}(A^\top)$ are orthogonal in $\mathbb{R}^n$.
- $\mathbf{N}(A^\top)$ and $\mathbf{R}(A^\top) = \mathbf{C}(A)$ are orthogonal in $\mathbb{R}^m$.

Proof. $\mathbf{v} \in \mathbf{N}(A) \Leftrightarrow A\mathbf{v} = \mathbf{0}$. $\quad$ $\mathbf{w} \in \mathbf{C}(A^\top) \Leftrightarrow \mathbf{w} = A^\top \mathbf{x}$.
Then

$$\mathbf{v}^\top \mathbf{w} = \mathbf{v}^\top (A^\top \mathbf{x}) \stackrel{(1.4.1)}{=} (\mathbf{v}^\top A^\top)\mathbf{x} \stackrel{(2.4.2)}{=} \underbrace{(A\mathbf{v})^\top}_{\mathbf{0}^\top} \mathbf{x} = 0.$$

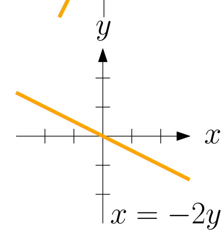
Same for $\mathbf{N}(A^\top)$ and $\mathbf{C}(A)$.

$\mathbf{R}\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right)$



$\mathbf{N}\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right)$

(3.2)

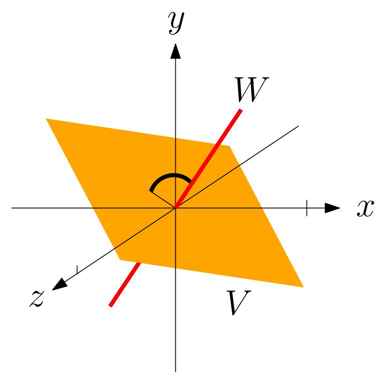


Exercise: If V and W are orthogonal, $V \cap W = \{\mathbf{0}\}$ (only the zero vector is in both).

If V and W are subspaces of $\mathbb{R}^n$ such that $V \cap W = \{\mathbf{0}\}$, then $\dim(V) + \dim(W) \leq n$.

Proof. Let $k = \dim(V), \ell = \dim(W)$, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ a basis of V, $\mathbf{w}_1, \ldots, \mathbf{w}_\ell$ a basis of W. Want to show: these $k + \ell$ vectors are independent.

Suppose $\underbrace{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k}_{\mathbf{v} \in V} + \underbrace{d_1\mathbf{w}_1 + \cdots d_\ell\mathbf{w}_\ell}_{\mathbf{w} \in W \ (\Rightarrow -\mathbf{w} \in W)} = \mathbf{0}$. Then $\mathbf{v} = -\mathbf{w} \in V \cap W$, so $\mathbf{v} = \mathbf{w} = \mathbf{0}$. $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and $\mathbf{w}_1, \ldots, \mathbf{w}_\ell$ are independent $\Rightarrow c_1, \ldots, c_k = 0$ and $d_1, \ldots, d_\ell = 0 \Rightarrow \mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell$ are independent (1.3.3) $\Rightarrow k + \ell \leq n$.



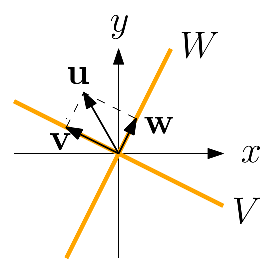
$\dim(V) = 2, \ \dim(W) = 1$

4.1.1 Orthogonal complement $V^\perp$

V subspace of $\mathbb{R}^n$.
Definition: $\mathbf{w} \in \mathbb{R}^n$ is orthogonal to V if $\mathbf{w}$ is orthogonal to all vectors in V.
$V^\perp$: all vectors in $\mathbb{R}^n$ that are orthogonal to V. Exercise: $V^\perp$ is a subspace.

Let V, W be orthogonal subspaces of $\mathbb{R}^n$. The following statements are equivalent.

		$V = \mathbf{N}(A), W = \mathbf{R}(A) = \mathbf{C}(A^\top)$
(i)	$W = V^\perp$	true
		$\Uparrow$
(ii)	$\dim(V) + \dim(W) = n$	true: $(n-r) + r = n$ (3.5)
		$\Downarrow$
(iii)	every $\mathbf{u} \in \mathbb{R}^n$ can be written as $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with *unique* vectors $\mathbf{v} \in V, \mathbf{w} \in W$	true



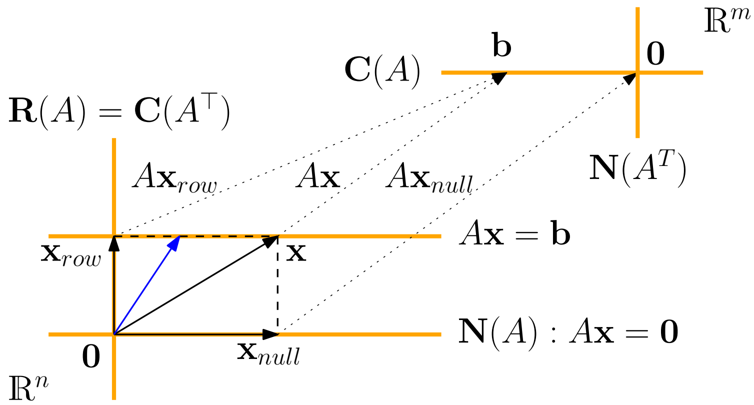
Proof: $\mathbf{v}_1, \ldots, \mathbf{v}_k$ a basis of V, $\mathbf{w}_1, \ldots, \mathbf{w}_\ell$ a basis of W.
(i)$\Rightarrow$(ii): Observation: $\mathbf{w} \in \mathbb{R}^n$ orthogonal to $V \Leftrightarrow \mathbf{w}$ orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Let A be the matrix with rows $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then $V = \mathbf{C}(A^\top)$ (dimension k) and $W = V^\perp = \mathbf{N}(A)$ (dimension $n - k$, 3.5).
(ii)$\Rightarrow$(iii): As previously seen, $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell$ are independent. Since $k + \ell = n$, they are a basis of $\mathbb{R}^n$. So

$$\mathbf{u} = \underbrace{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k}_{\mathbf{v}} + \underbrace{d_1\mathbf{w}_1 + \cdots d_\ell\mathbf{w}_\ell}_{\mathbf{w}}$$

with unique scalars (3.4) $\Rightarrow$ unique $\mathbf{v}, \mathbf{w}$.
(iii)$\Rightarrow$(i): We need that W contains *all* vectors orthogonal to V. Let $\mathbf{u} \in \mathbb{R}^n$ be orthogonal to V. We can write $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{v} \in V, \mathbf{w} \in W$. Multiplying with $\mathbf{v}$ from the left,

$$\underbrace{\mathbf{v}^\top\mathbf{u}}_{0} = \mathbf{v}^\top\mathbf{v} + \underbrace{\mathbf{v}^\top\mathbf{w}}_{0} \quad \Rightarrow \quad \mathbf{v}^\top\mathbf{v} = \|\mathbf{v}\|^2 = 0 \quad \Rightarrow \quad \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{u} = \mathbf{w} \in W.$$

4.1.2 The big picture



(3.3): Solutions of $Ax = b$ $=$ particular solution of $A\mathbf{x} = \mathbf{b}$ $+$ solutions of $A\mathbf{x} = \mathbf{0}$

(4.1): $\mathbf{N}(A)$ and $\mathbf{C}(A^\top)$, $\mathbf{N}(A^\top)$ and $\mathbf{C}(A)$ are orthogonal subspaces...

(4.1.1): ...and orthogonal complements. For $\mathbf{x} \in \mathbb{R}^n$: $\mathbf{x} = \mathbf{x}_{row} + \mathbf{x}_{null}$ (row space and nullspace components). If $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_{row} = \mathbf{b}$, $A\mathbf{x}_{null} = \mathbf{0}$.

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