CS Lens

Graphs, Networks, and Linear Algebra

(Part 1)

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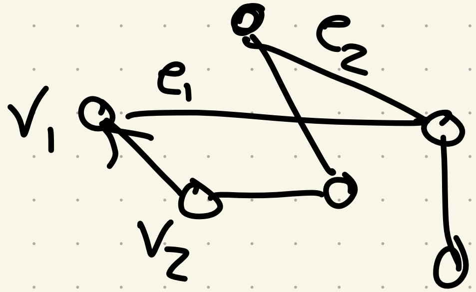
CS Lecs:



G a graph

$G = (V, E)$ $\quad |V| = n$

$|E| = m$

$E \subseteq \binom{V}{2}$

B $\quad n \times m$ $\quad$ incidence matrix

$$B \qquad \begin{array}{c c} & \begin{array}{ccc} e_1 & \cdots & e_m \end{array} \\ \begin{array}{c} v_1 \\ \vdots \\ \vdots \\ v_n \end{array} & \begin{bmatrix} 1 & & \\ & & \\ -1 & & \\ & & \end{bmatrix} \end{array}$$

$$B_{ij} = \begin{cases} -1 & \text{if } e_j \text{ leaves } v_i \\ 1 & \text{if } e_i \text{ arrives } v_i \\ 0 & \text{o.w.} \end{cases}$$

What is $N(B^T)$?

$\left(x \in N(B^T) \quad B^T x = 0 \right)$

$B^T x = 0$

for each edge e_j

$$(B^T x)_j = X_{\substack{\text{dst} \\ \text{of } e_j}} - X_{\substack{\text{orig.} \\ \text{of } e_j}}$$

If $x \in N(B^T)$ if v_i is connected to v_j

then $X_i = X_j$.

If there is a path between every pair of nodes we say G is connected.

$$x = \alpha \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \text{ is in } N(B^T) \ \forall_{\alpha \in \mathbb{R}}$$

Theorem: $\text{Dim}(N(B^T)) = 1$ iff G is a connected graph

In fact, $\text{Dim}(N(B^T))$ "counts" # of connected components (Try to prove it)