Linear Algebra

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Linear Programming

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Linear equations vs. Linear inequalities

Problem: Solve $A\mathbf{x} = \mathbf{b}$!

Algorithm: Gauss-Jordan elimination ($A = CR$).

Problem: Solve $A\mathbf{x} \leq \mathbf{b}$!

Example:

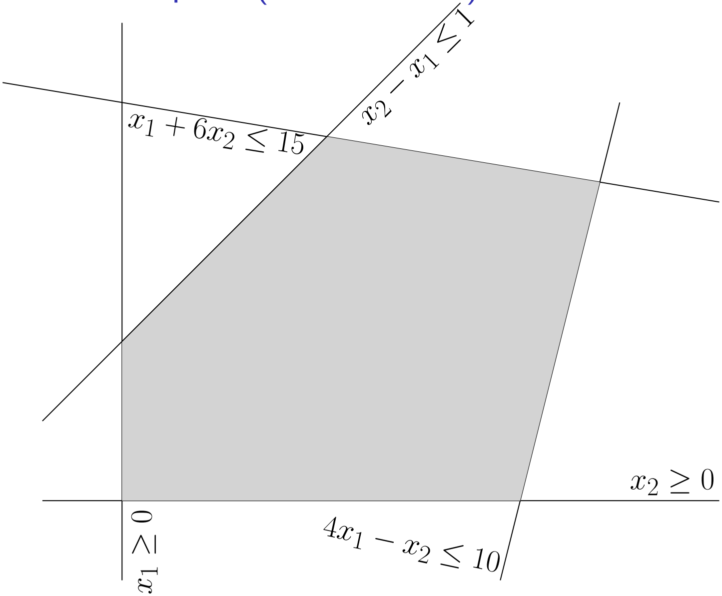
$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_2 - x_1 &\leq 1 \\ x_1 + 6x_2 &\leq 15 \\ 4x_1 - x_2 &\leq 10 \end{aligned}$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 1 \\ 1 & 6 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 15 \\ 10 \end{bmatrix}$$

This is called **Linear Programming** and has many important applications [MG07].

Algorithm: ?

The solution space (subset of $\mathbb{R}^2$)



Intersection of five **halfplanes**

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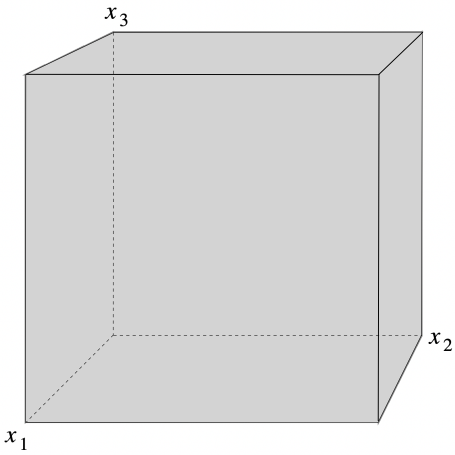
Halfplane: everything on one side of a line, including the line

The solution space (subset of $\mathbb{R}^3$)

$$\begin{array}{ccccc} 0 & \leq & x_1 & \leq & 1 \\ 0 & \leq & x_2 & \leq & 1 \\ 0 & \leq & x_3 & \leq & 1 \end{array}$$

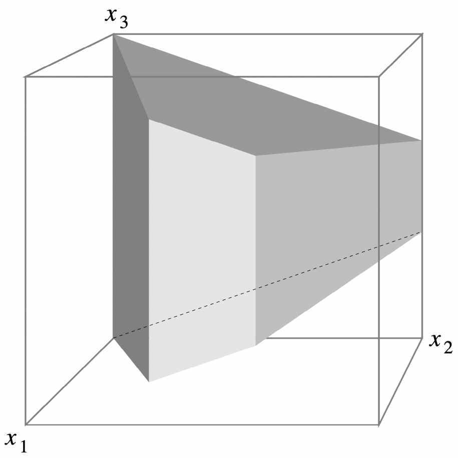
Intersection of six **halfspaces**



Unit cube

The solution space (subset of $\mathbb{R}^3$)

$$
\begin{array}{rcccl}
0 & \leq & x_1 & \leq & 1 \\
\frac{1}{3}x_1 & \leq & x_2 & \leq & 1 - \frac{1}{3}x_1 \\
\frac{1}{3}x_2 & \leq & x_3 & \leq & 1 - \frac{1}{3}x_2
\end{array}
$$



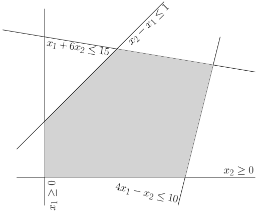
Klee-Minty Cube

The solution space (subset of $\mathbb{R}^n$)

A an $m \times n$ matrix, $b \in \mathbb{R}^m$

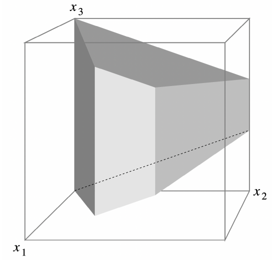
Solutions of $A\mathbf{x} \leq \mathbf{b}$: intersection of m halfspaces in $\mathbb{R}^n$, a **convex polyhedron**

$\mathbb{R}^2$



Convex polygon

$\mathbb{R}^3$



Convex polyhedron

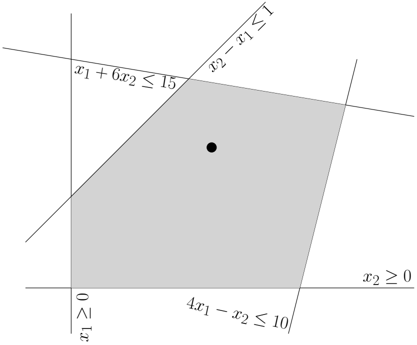
$\mathbb{R}^n$

Beast

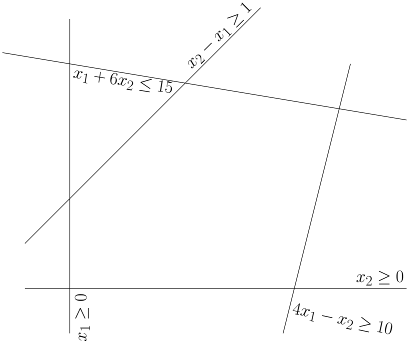
Linear inequalities, geometrically

Problem, algebraically: Solve $A\mathbf{x} \leq \mathbf{b}$!

Problem, geometrically: Find a point in a convex polyhedron, or conclude that it is empty!



point in convex polyhedron = solution

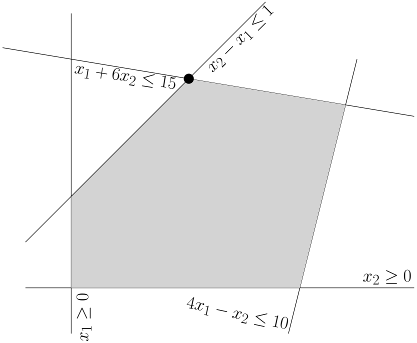


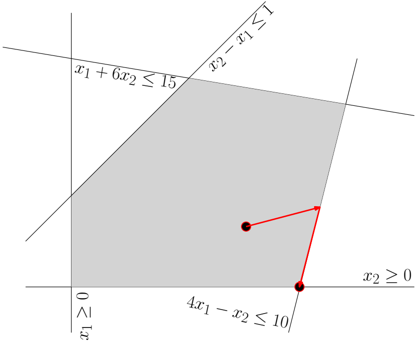
convex polyhedron empty, no solution

Finding a point in a convex polyhedron...

...is "the same" as finding a corner.[1]

If we have a corner, we have a point, and if we have a point, we can easily find a corner (walk until we hit a wall, walk inside the wall until we hit another wall,...)

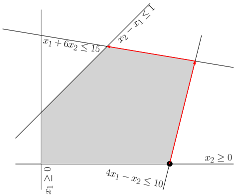




[1]For this, we need to assume that the convex polyhedron is bounded, but this is no problem.

Solving $A\mathbf{x} \le \mathbf{b}$

Consider the inequalities

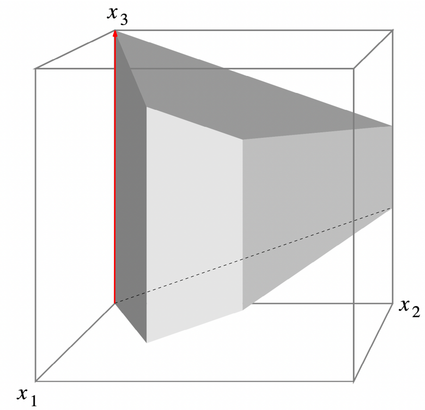


$$A\mathbf{x} + x_{n+1} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \le \mathbf{b}, \quad x_{n+1} \le 0.$$

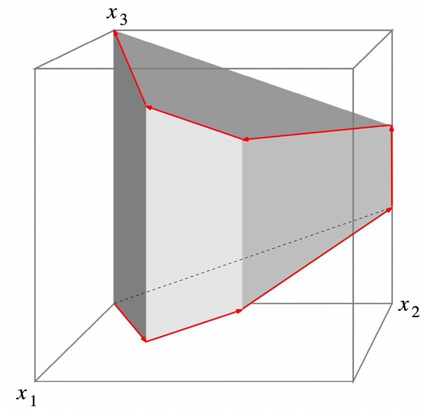
- The corresponding convex polyhedron is nonempty, and a point in it can be found easily (set $\mathbf{x} = \mathbf{0}$ and make x_{n+1} small enough).
- From this point, find a corner, as described before.
- **From this corner, "climb up" along edges to the highest corner (= highest point, the one with largest x_{n+1}-value).**
- If the highest corner has $x_{n+1} = 0$, we have solved $A\mathbf{x} \le \mathbf{b}$, otherwise, $A\mathbf{x} \le \mathbf{b}$ has no solution.

This is George Dantzig's **simplex method** from the 1940's [Dan63].

Depending on the climbing rule, the simplex method...



...can be very fast...



...or very slow.

n-dimensional Klee-Minty cube: a natural climbing rule visits all 2^n corners [KM72].

Simplex method = "Gauss elimination for linear inequalities"

- Extremely fast in practice
- But: For every climbing rule that people have developed, there are (artificially constructed) beasts on which climbing takes very long when this rule is used.

Open problem: Is there a climbing rule which climbs *every* beast quickly?

A positive answer would solve Smale's 9th problem for the 21st century:

`https://en.wikipedia.org/wiki/Smale%27s_problems`

- There are randomized climbing rules (using coin flips) which are (in expectation) faster than the known deterministic ones (no coin flips).
- One concrete result here: There is a rule that climbs every n-dimensional cube in at most $e^{2\sqrt{n}}$ steps in expectation (much better than the worst case 2^n) [Gär02].

References

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