Linear Algebra

8.11.2023

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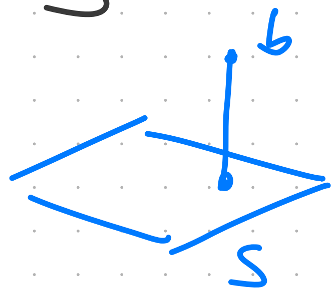
* Please take a look at the typed lecture notes

Question: What do we do when we have a linear system without solutions?

linear system $Ax = b$ but there is no x s.t. $Ax = b$.

e.g. too many equations.

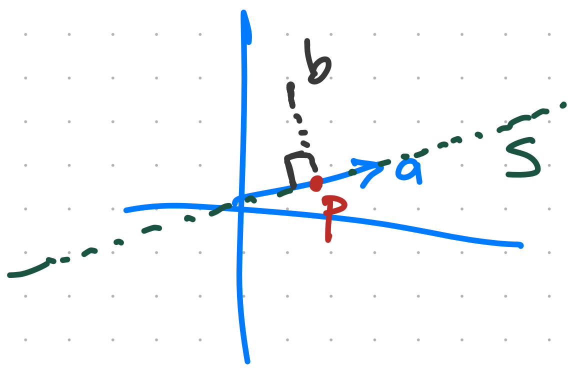
Projections: Given a subspace S and a vector b, we define the projection of b onto S as the closest point to b in S.



$$\mathrm{Proj}_S(b) = \underset{p \in S}{\mathrm{argmin}} \|b - p\|$$

Challenge: Show it exists and is unique.

Projection on a line



$$a \in \mathbb{R}^m \qquad a \neq 0$$

$$S = \mathrm{span}(a)$$
$$= \{\alpha \cdot a : \alpha \in \mathbb{R}\}$$

$p = \hat{x} a$, for some $\hat{x}$ $\qquad (e = b - p)$

and $\quad b - p \perp a$

$$\begin{pmatrix} a \cdot (b-p) = \\ = \langle a, b-p \rangle \end{pmatrix} a^T (b - p) = 0 \iff a^T (b - \hat{x} a) = 0$$

$$a^T b = \hat{x}\, a^T a \iff \hat{x} = \frac{a^T b}{a^T a} \Rightarrow p = \frac{a^T b}{a^T a}\, a$$

$\Rightarrow$

$$p = \frac{a a^T}{a^T a}\, b$$

Prop. For $a \in \mathbb{R}^m$, $a \neq 0$ the projection of b on $S = \mathrm{span}(a)$ is given by

$$\mathrm{Proj}_{\mathrm{Span}(a)}(b) = \frac{a a^T}{a^T a}\, b.$$

Sanity check: If $b = \beta a$ for $\beta \in \mathbb{R}$ it should be that $\mathrm{Proj}_{\mathrm{span}(a)}(\beta a) = \beta a$.

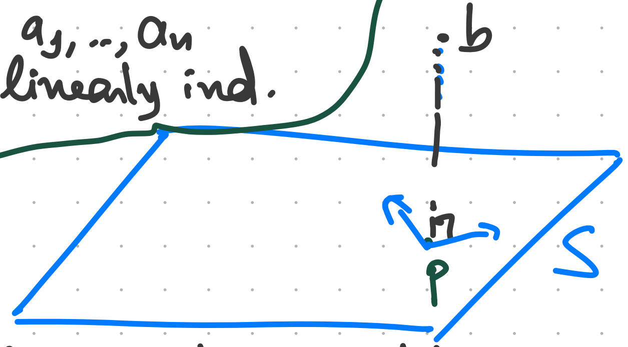
Indeed $\quad \dfrac{a a^T}{a^T a}\, \beta a = \beta\, \dfrac{a \cancel{a^T a}}{\cancel{a^T a}} = \beta a$ ✓

Let's do general subspaces.

S subspace in $\mathbb{R}^m$ with dim $n > 0$.

Let $a_1, \dots, a_n$ be a basis of S, i.e. $S = \mathrm{Span}(a_1, \dots, a_n)$

$S = C(A)$ where $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$ and $a_1, \dots, a_n$ are linearly ind.



The projection p of b on S should be a vector p s.t.

$$b - p \perp S, \quad b - p \perp a \quad \forall a \in S$$

$b - p \perp a_k$ for each $k = 1, \dots, n$.

Detour:

Prop: $x \in \mathbb{R}^m$ is orthogonal to all $a \in S$ iff $x \perp a_k$ for $k = 1, \dots, n$ where $a_1, \dots, a_n$ is a basis of S.

Proof: $\Rightarrow$ trivial.

$\Leftarrow$ interesting

Let $a \in S$, because $a_1, \dots, a_n$ is a basis of S then $a = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n$ for some $\beta_1, \dots, \beta_n \in \mathbb{R}$

but then

$$\begin{aligned} x^T a &= x^T \beta_1 a_1 + x^T \beta_2 a_2 + \dots \\ &= \beta_1 x^T a_1 + \beta_2 x^T a_2 + \dots \\ &= 0. \end{aligned}$$

$\square$.

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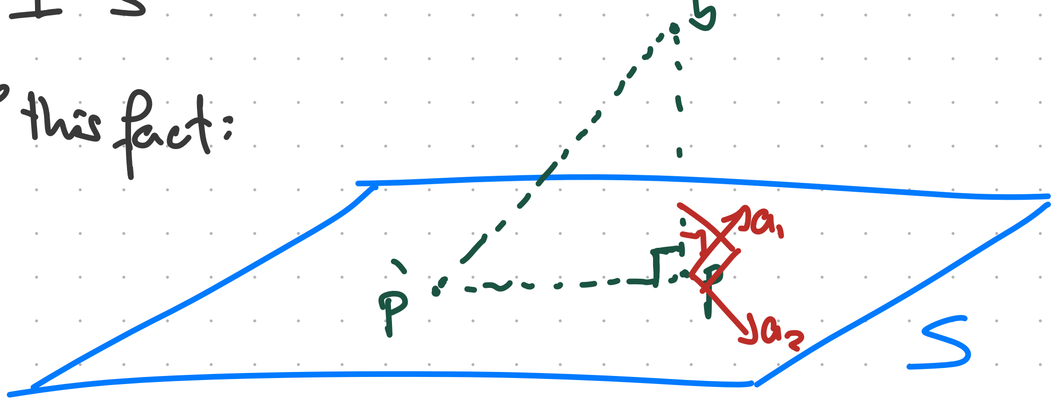
The projection p of b on S is the point p s.t. $b - p \perp S$

Proof of this fact:



Since $p, p' \in S$, $p' - p \in S$ and so $p' - p \perp b - p$

by Pythagoras $\|p' - b\|^2 = \|b - p\|^2 + \underbrace{\|p' - p\|^2}_{\geq 0 \text{ and } = 0 \text{ iff } p' = p.}$

and so $\|p' - b\| \geq \|b - p\|$

and $=$ iff $p = p'$.

p is the vector is S s.t. $b - p \perp a_k$ for $k = 1, \dots, n$

$$a_k^T(b-p) = 0 \text{ for } k = 1, \dots, n$$

since $p \in S$, $p = \hat{x}_1 a_1 + \hat{x}_2 a_2 + \cdots + \hat{x}_n a_n$

$$p = A\hat{x}, \quad A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \end{bmatrix}$$

$p = A\hat{x}$ and

$$a_k^T(b - A\hat{x}) = 0 \text{ for } k = 1, \dots, n$$

$$\Leftrightarrow \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_n^T & - \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (=0)$$

$$\Leftrightarrow A^T(b - A\hat{x}) = 0$$

$$\Rightarrow A^T A \hat{x} = A^T b \quad \text{(normal equations)}$$

If $A^T A$ were to be invertible, then

$$\hat{x} = (A^T A)^{-1} A^T b$$

Is $A^T A$ invertible?

$$p = A\hat{x} = A(A^T A)^{-1} A^T b$$

Proposition (4.2.4) A^TA is invertible iff A has linearly ind. columns.

A is $m \times n$ $(m \geq n)$
A^T is $n \times m$
A^TA is $n \times n$

Proof: We will show that A^TA and A have the same Nullspace $(N(A^TA) = N(A))$

A has ind. columns $\iff N(A) = \{0\}$

since A^TA is a square matrix: A^TA is invertible $\iff N(A^TA) = \{0\}$.

- If $x \in N(A)$ then $Ax = 0$ and so $A^TAx = 0$ thus $x \in N(A^TA)$

so $N(A) \subseteq N(A^TA)$

- If $x \in N(A^TA)$ then $A^TAx = 0$ thus $x^TA^TAx = x^T0 = 0$

but $x^TA^TAx = (Ax)^T(Ax) = \|Ax\|^2$

so $\|Ax\| = 0$ which implies $Ax = 0$. Hence $x \in N(A)$. $\square$

Theorem: Let S be a subspace of $\mathbb{R}^m$ with a basis $a_1, \dots, a_n$ $(n \geq 1)$. Let $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$. For $b \in \mathbb{R}^m$

$$\text{Proj}_S(b) = A(A^TA)^{-1}A^Tb$$

* agrees with $n = 1$
* If $b \in S$ $\text{proj}_S(b) = b$ (HW)

$$\text{Proj}_S(b) = Pb, \quad P = A(A^TA)^{-1}A^T$$

* P^2 ? we should have $P^2 = P$

so let's check

$$P^2 = A\underbrace{(A^TA)^{-1}A^TA}_{=I}(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T \checkmark$$

HW: What is $I - P$?

HW: why can't we "expand" $A(A^TA)^{-1}A^T$ $\neq A A^{-1}A^{T^{-1}}A^T \neq I$?