# Linear Algebra

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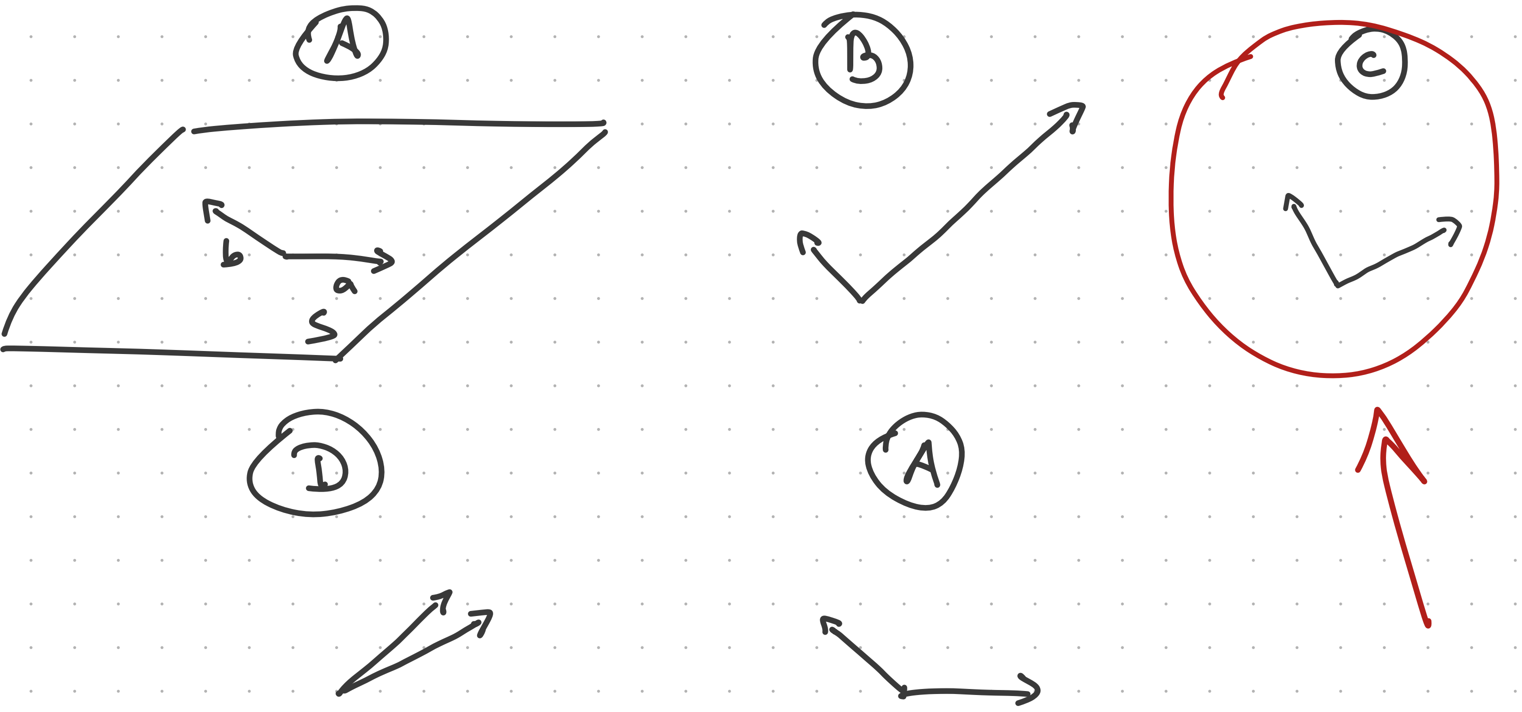
* Please take a look at the typed lecture notes




## (4.4.) Orthonormal Bases and Gram Schmidt

**Definition** (Orthonormal basis)

$q_1, \dots, q_n$ a basis of $S$ (subspace $\mathbb{R}^m$) is called an orthonormal basis iff

$$\forall_{i,j} \quad 1 \le i, j \le n$$

$$q_i^T q_j = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w} \end{cases}$$

in other words
if $i \neq j$
$q_i^T q_j = 0 \quad q_i \perp q_j$
and
$q_i^T q_i = 1 \quad \|q_i\|^2 = 1$

In matrix notation $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$ $m \times n$ $(n \le m)$

$$(Q^T Q)_{ij} = q_i^T q_j = \delta_{ij}$$

$$Q^T Q = I$$

Example (4.4.2)

The canonical basis for $\mathbb{R}^m$ is an example of an orthonormal basis.

$e_1, \ldots, e_m$ where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ entry $i$, $(e_i)_j = \delta_{ij}$.

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$q_1, \ldots, q_n$ are an orthonormal basis for a subspace $S$ of $\mathbb{R}^m$

$Q \qquad Q^T Q = I$

what is $QQ^T$?

if $n < m$ $(S \neq \mathbb{R}^m)$

$$\begin{array}{cc} & n \\ Q = & \begin{bmatrix} & \\ & \\ & \end{bmatrix} \\ m & \end{array}$$

$Q^T Q = \begin{bmatrix} & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}$

$QQ^T = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \end{bmatrix}$

$\neq I$.

<u>Caution:</u> $QQ^T$ is not necessarily $= I$.

If $n = m$ $(S = \mathbb{R}^m)$

then $Q^T Q = I \Rightarrow Q^T = Q^{-1}$

$QQ^T = I$.

**Def (4.4.3)** An $n\times n$ matrix $Q$ satisfying $Q^TQ = I$ is called an orthogonal matrix.

Some $2\times 2$ examples

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \qquad \tilde{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

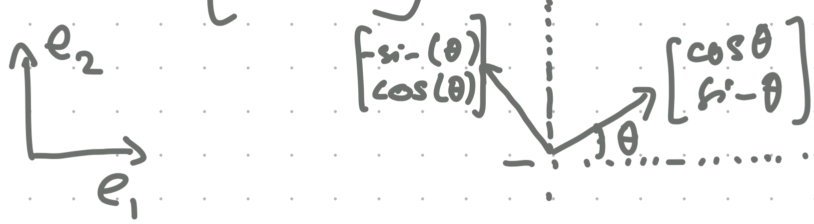
$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \checkmark \qquad \tilde{Q}^T\tilde{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$\tilde{Q}$ is not orthogonal.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_\theta^T R_\theta = \begin{bmatrix} \cos^2+\sin^2 & 0 \\ 0 & \cos^2+\sin^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_\theta e_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad R_\theta e_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

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**Prop 4.4.7** If $Q$ is an orthogonal matrix ($n\times n$) then $\forall x, y \in \mathbb{R}^n$

$$\|Qx\| = \|x\|$$

$$(Qx)^T(Qy) = x^Ty$$

Proof: $\|Qx\| = \|x\| \iff \|Qx\|^2 = \|x\|^2$

$$\|Qx\|^2 = (Qx)^T(Qx) = x^T \underbrace{Q^T Q}_{=I} x = x^T x = \|x\|^2$$

$$(Qx)^T(Qy) = x^T Q^T Q y = x^T y \quad \square.$$

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example:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c \\ a \\ d \\ b \end{bmatrix}$$

$$\overset{\parallel}{P}$$

Challenge: show that any permutation matrix is an orthogonal matrix.

Challenge (☆): Show that for any $P$ permutation matrix there exists $k$ a positive integer such that $P^k = I$.

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## 4.4.1 Projection with orthonormal basis

**Prop:** Let $S$ be a subspace of $\mathbb{R}^m$ and $q_1, \dots, q_n$ an orthonormal basis of $S$.

$$Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

the projection Matrix that projects to $S$ $(C(Q))$ is given by $QQ^T$.

**Proof:** The projection matrix is given by

$$P = Q\underbrace{(Q^TQ)}_{I}{}^{-1}Q^T = QQ^T.$$

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$$\mathrm{Proj}_S(b) = QQ^Tb.$$

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What is the Least Squares solution for

$$Qx = b$$

we know it is the solution to the normal eq.

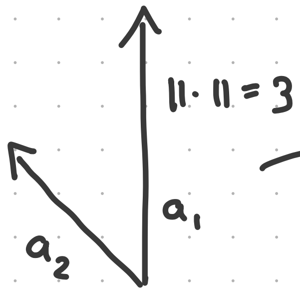
$$Q^TQ\hat{x} = Q^Tb$$

$$\hat{x} = Q^Tb$$

How do we build Orthonormal basis?

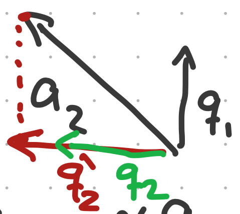
let $a_1, \dots, a_n$ be a basis of $S$ (subspace of $\mathbb{R}^m$)

how do we go from $a_1, \to a_n$ to an orthonormal basis $q_1, \dots, q_n$?




$$\begin{bmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{bmatrix}$$

$$q_1 = \frac{a_1}{\|a_1\|}, \quad \|q_1\| = 1$$





$$q_2' = a_2 - \alpha q_1 \quad , \; q_2'^T q_1 = 0$$

$$(a_2 - \alpha q_1)^T q_1 = 0$$

$$a_2^T q_1 - \alpha = 0$$

$$\alpha = (a_2^T q_1)$$

$$q_2' = a_2 - (a_2^T q_1) q_1$$

$$q_2 = q_2' / \|q_2'\|$$

$$q_1 = \frac{a_1}{\|a_1\|} \quad \text{and} \quad q_2 = \frac{a_2 - (a_2^T q_1) q_1}{\|a_2 - (a_2^T q_1) q_1\|}$$

4.4.2. Gram - Schmidt Process:

**Algorithm 4.4-10.**

Given $n$ linearly independent vectors $a_1, \ldots, a_n$ (they are a basis for $C(A)$) the Gram - Schmidt Process computes an orthonormal basis $q_1, \ldots, q_n$ of $C(A)$ in the following way

$$q_1 = a_1 / \|a_1\|$$

for $k = 2, \ldots, n$ do

$$q_k = \frac{a_k - \mathrm{Proj}_{\mathrm{Span}(q_1, \ldots, q_{k-1})}(a_k)}{\|a_k - \mathrm{Proj}_{\mathrm{Span}(\ldots)}(a_k)\|}$$

$$\mathrm{Proj}_{\mathrm{Span}(q_1, \ldots, q_{k-1})}(a_k) = (a_k^T q_1) q_1 + (a_k^T q_2) q_2 + \cdots + (a_k^T q_{k-1}) q_{k-1}$$

$$= \sum_{j=1}^{k-1} (a_k^T q_j) q_j$$

Sanity check: $q_k^T q_i = 0$ for $i < k$

$$\left(a_k - \sum_{j=1}^{k-1} (a_k^T q_j) q_j\right)^T q_i = a_k^T q_i - \sum_{j=1}^{k-1} (a_k^T q_j)(q_j^T q_i)$$

Proof: Try to do it (detailed in notes but try before ready)

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Let $A$ be an $m \times n$ matrix with ind. columns.

$Q$ the $m \times n$ matrix whose columns are the orthonormal basis outputted by Gram-Schmidt.

$C(A) = C(Q)$ $\qquad$ $Q^T Q = I$.

$R := Q^T A$ $\qquad$ what do we know about $R$?

$R_{ij}$ $\quad i > j$. $\quad R_{ij} = q_i^T a_j = 0$

$R_{21}$ $\quad q_2^T a_1 = 0$ ?