Linear Algebra

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* Please take a look at the typed lecture notes

4.4.2. Gram - Schmidt Process:

From last lecture

Algorithm 4.4.10.

Given n linearly independent vectors $a_1, \dots, a_n$ (they are a basis for $C(A)$) the Gram - Schmidt Process computes an orthonormal basis $q_1, \dots, q_n$ of $C(A)$ in the following way

$$q_1 = \frac{a_1}{\|a_1\|}$$

for $k = 2, \dots, n$ do

$$q_k = \frac{a_k - \mathrm{Proj}_{\mathrm{Span}(q_1, \dots, q_{k-1})}(a_k)}{\|a_k - \mathrm{Proj}_{\mathrm{Span}(\dots)}(a_k)\|}$$

$$\mathrm{Span}(q_1, \dots, q_{k-1}) = \mathrm{Span}(a_1, \dots, a_{k-1})$$

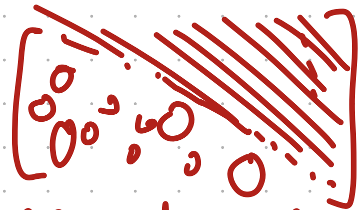
Let A $m \times n$ with ind columns

Let Q $m \times n$ whose columns are the output of Gram - Schmidt on $a_1, \dots, a_n$

$$C(Q) = C(A)$$

$R := Q^T A$ $\qquad$ R is $n \times n$

$R_{ij} = 0$ if $i > j$.

$R =$ 

upper triangular

$R_{ij} = q_i^T a_j$

$q_i \perp \text{Span}(a_1, \dots, a_{i-1})$

$q_i \perp a_j$

$R = Q^T A$

$QR = QQ^T A$

$= A$

QQ^T is the proj on $C(Q)$

but $C(A) = C(Q)$

QQ^T is proj on $C(A)$.

(Def: 4.4.12)

QR-decomposition.

A $m \times n$ with ind columns.

$$A = QR$$

$Q^T Q = I$

R is upper triangular.

Fact 4.4.13.

A $m \times n$ with ind columns. $\qquad$ $A = QR$

$C(A) = C(Q)$

Proj on $C(A)$ is the same as proj on $C(Q)$

$$\text{Proj}_{C(A)}(b) = QQ^T b$$

→ Least Squares of $Ax = b$

is the solution to $A^TA\hat{x} = A^Tb$

$A = QR$

$$\Leftrightarrow$$

$$(QR)^T(QR)\hat{x} = (QR)^Tb$$

$$R^T\underbrace{Q^TQ}_{=I}R\hat{x} = R^TQ^Tb$$

what is $N(R)$?

if $x \in N(R)$

$QRx = 0$

$Ax = 0$

$x \in N(A)$

$\Rightarrow x = 0$

$$R^TR\hat{x} = R^TQ^Tb$$

$$R\hat{x} = Q^Tb$$

(since R^T is inv)

so $N(R) = \{0\}$

R is inv

R^T is inv

can be solved "backsubstitution" since R is up. triangular

Guiding Question: How do we "invert" a matrix that's not invertible?

Goal: Given a matrix A $m \times n$ we want to define/construct A^+ pseudo-inverse of A.

$$A : \underset{\in \mathbb{R}^n}{x} \longrightarrow \underset{\in \mathbb{R}^m}{Ax} = b$$

Goal $A^+ : b \longrightarrow x$???

why not possible:
(i) maybe for some $b \in \mathbb{R}^m$ there is no x s.t. $Ax = b$
(ii) for some $b \in \mathbb{R}^m$ there may be many $x \in \mathbb{R}^n$ such that $Ax = b$
which one do we pick?
(iii) should be of the form $\hat{x} = A^+ b$
matrix.

Def 4.5.1 $A \in \mathbb{R}^{m \times n}$ ind columns $\Leftrightarrow$ rank$(A) = n$
$\Leftrightarrow$ full column rank

$[A]$

$$A^+ = (A^TA)^{-1}A^T$$

A^+ takes b to the least squares sol. of $Ax = b$

Prop: $A^+A = I$

$[\ A^+\ ][A] = [I]$

$[A][\ A^+\ ] \neq I$

Proof:
$$A^+A = (A^TA)^{-1}A^TA$$
$$= I \quad ✓$$

A is $m \times n$ $\quad$ A has ind rows

full row rank

$\operatorname{rank}(A) = m$

$[\qquad]$

$m \leq n$

we can define

$$A^+ = \left((A^T)^+\right)^T =$$

$$= \left(((A^T)^T A^T)^{-1} (A^T)\right)^T$$

$$= \left((AA^T)^{-1} A\right)^T = A^T\left((AA^T)^{-1}\right)^T$$

$$= A^T\left((AA^T)^T\right)^{-1} = A^T(AA^T)^{-1}.$$

Def: A $m \times n$ with $\operatorname{rank}(A) = m$

$$A^+ := A^T(AA^T)^{-1}.$$

Prop: if A is full row rank

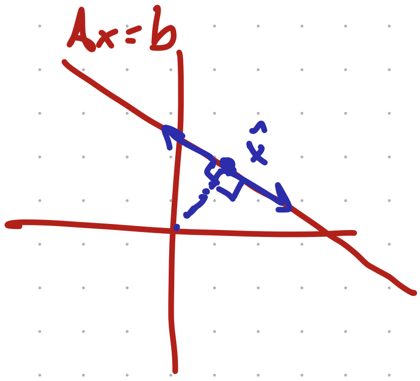
$AA^+ = I$.

Proof: $AA^T(AA^T)^{-1} = I$.

$A \quad m \times n \qquad \text{rank}(A) = m$

for any $b \in \mathbb{R}^m$ $\quad Ax = b$ has a solution x.

Any $x \in N(A)$ is a solution to $Ax = 0$.



$$\begin{bmatrix} & & \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = 0$$

<u>natural:</u> $\hat{x}$ to be the solution of

$$\min \|x\|^2 \quad \text{s.t. } Ax = b \qquad (10)$$

both x_1 and x_2 are solutions of $Ax=b$

$$A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0 \Rightarrow x_1 - x_2 \in N(A)$$

if $y \in N(A)$ and $Ax = b$ then $A(x+y) = Ax + Ay = b$

given x_0 s.t. $Ax_0 = b$ all solutions of $Ax = b$ are of the form $x_0 + y$ for $y \in N(A)$.

(Careful proof on the notes)

$\hat{x}$ unique sol of (10) is $\hat{x}$ s.t.

$Ax = b$ and $x \in N(A)^\perp = C(A^T)$

Prop: for A full row rank

$\hat{x} = A^+ b$ is the unique sol of (10).

Proof: $\rightarrow A\hat{x} = AA^+b = b$ ✓

need to show

$\hat{x} \in N(A)^\perp \iff \hat{x} \in C(A^T)$

$\hat{x} = A^+b = \underline{A^T}(AA^T)^{-1} b \in C(A^T)$ ✓

What if A is neither full row or column rank?

idea: $\hat{x} = A^+b$ to be sol of $\min \|x\|$ s.t. $A^TAx = A^Tb$