Linear Algebra

22.11.2023

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* Please take a look at the typed lecture notes

From last lecture

Def 4.5.1 $A \in \mathbb{R}^{m \times n}$ ind columns $\Leftrightarrow$ rank$(A) = n$

$\Leftrightarrow$ full column rank

$[A]$

$$A^+ = (A^T A)^{-1} A^T$$

A^+ takes b to the least squares sol. of $Ax = b$

Def: A $m \times n$ with rank$(A) = m$

$$A^+ := A^T (A A^T)^{-1}.$$

$[\ A\ ]$

A is full row rank

How to remember?

If A is not inv. at most one of $A^T A$ or $A A^T$ can be invertible.

What if A is neither full row or column rank?

idea: $\hat{x} = A^+ b$ to be sol of $\min \|x\|$ s.t. $A^TAx = A^Tb$ (11)

for example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$Ax = b$$

Def 4.5.7

A $m \times n$, $r = \text{rank}(A)$, $r = \dim C(A)$

$A = CR$ decomposition

C $m \times r$ matrix with the first r lin. ind. columns of A

columns of C are a basis for $C(A)$.

every column of A (a_i) can be written as a linear combination of columns of C. $a_i = C r_i$

$a_i \in \mathbb{R}^m$, $C \in \mathbb{R}^{m \times r}$, $r_i \in \mathbb{R}^r$

$$R = \begin{bmatrix} | & | & & | \\ r_1 & r_2 & \cdots & r_n \\ | & | & & | \end{bmatrix}$$

$A = CR$ $\quad A \in \mathbb{R}^{m \times n}$ $\quad r = \text{rank}(A)$

$r < n$

$r < m$

$$\begin{array}{c c} & n \\ m & \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \end{array} = \begin{array}{c c} & r \\ m & \begin{bmatrix} \\ C \\ \\ \end{bmatrix} \end{array} \begin{array}{c c} & n \\ r & \begin{bmatrix} R \end{bmatrix} \end{array}$$

$\text{rank}(C) = r$ full column rank

$\text{rank}(R) = r$. full row rank.

$$A^+ = (CR)^+ = R^+ C^+$$

rxm $\left(C^+ = (\underbrace{C^TC}_{r\times r})^{-1} C^T \right.$

nxr $\left(R^+ = R^T (\underbrace{RR^T}_{r\times r})^{-1} \right.$

C^T rxm

R^T nxr

$$A^+ = R^T (RR^T)^{-1} (C^TC)^{-1} C^T$$

$$A^+ = R^T (C^T \underbrace{CR}_{A} R^T)^{-1} C^T$$

$$A^+ = R^T (C^T A R^T)^{-1} C^T .$$

Caution

$A = CR$ Not

$A = QR$ the

same R!!

Prop 4.5.8. $\hat{x} = A^+ b$ is the solution of (11).

(i) $\rightarrow$ $\hat{x}$ satisfies $A^TA\hat{x} = A^Tb$

(ii) $\rightarrow$ $\hat{x} \perp N(A^TA)$.

(i) $A = CR$

$$A^TA A^+ b = R^T \underbrace{C^T \underbrace{CR}_{A} R^T}_{} \underbrace{(C^TAR^T)^{-1}}_{} C^T b$$

I

$$= R^T C^T b = A^T b \checkmark$$

(ii) $\hat{x} \perp N(A^TA) \Leftrightarrow \hat{x} \in C((A^TA)^T) = C(A^TA)$

$= C(A^T)$

$\hat{x} \in C(A^T)$

$\hat{x} = R^T \underbrace{(C^TAR^T)^{-1} C^T b}_{}$, $\hat{x} \in C(R^T)$ $A^T = R^T C^T$

we need to show $\hat{x} \in C(A^T)$

we know $\hat{x} \in C(R^T)$

claim $C(A^T) = C(R^T)$

$\square$

$A^T = R^T C^T$

If $y \in C(A^T)$, $y = R^T \underbrace{C^T z}$

$y \in C(R^T)$

$C(A^T) \subseteq C(R^T)$

but they both have dim r.

so they are the same

The only thing we used about $A = CR$ was that C is full col. rank and R is full row rank.

Prop: $A \in \mathbb{R}^{m \times n}$, rank$(A) = r$.

Let $S \in \mathbb{R}^{m \times r}$, rank$(S) = r$, and $T \in \mathbb{R}^{r \times n}$ rank$(T) = r$

s.t.

$$A = ST$$

then

$$A^+ = T^+ S^+.$$

$\square$

Prop: for any matrices A, B. (s.t. AB makes sense)

(i) $(AB)^+ = B^+ A^+$

(ii) $(A^T)^+ = (A^+)^T$

→ (iii) AA^+ is symmetric is the proj matrix for $C(A)$

→ (iv) $A^+ A$ is symmetric. is the proj matrix for $C(A^T)$.

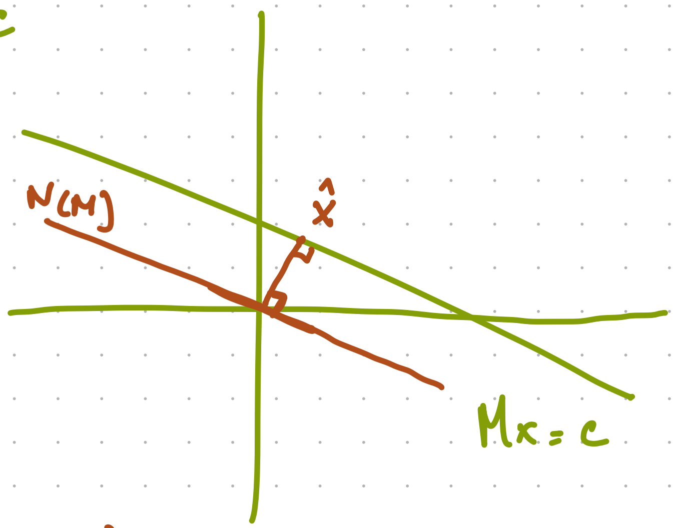
Proof: HW.

Clicker Question !

sol to $\min \|x\|$ s.t. $\underbrace{A^TA}_{M} x = \underbrace{A^Tb}_{c}$ is in $N(A^TA)^{\perp}$.

$\min \|x\|$

s.t. $Mx = c$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$$

$A^+ =$ $(A^TA)^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^+ = (A^TA)^{-1}A^T$$

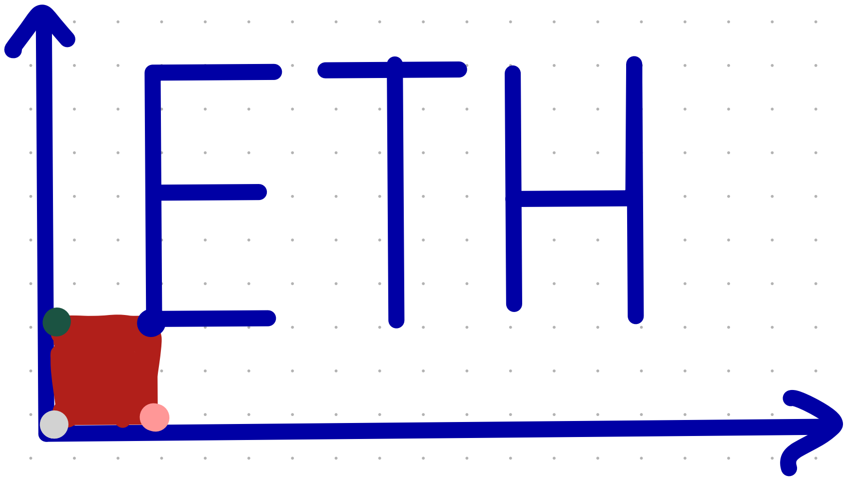
$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

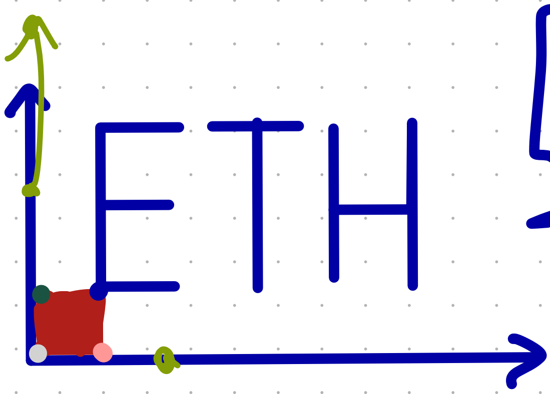
$$A^{+}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AA^{+} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

2×2 matrices

$A : x \longrightarrow Ax$

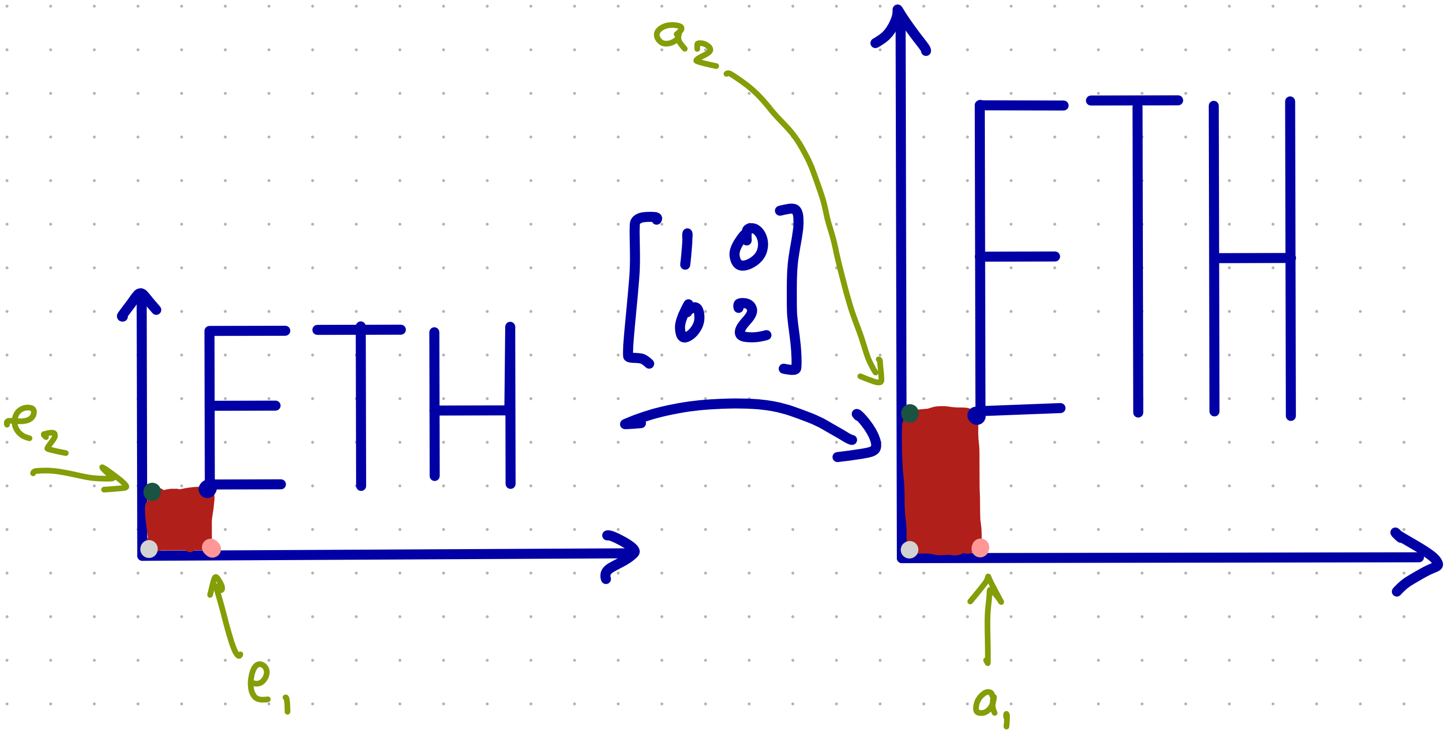




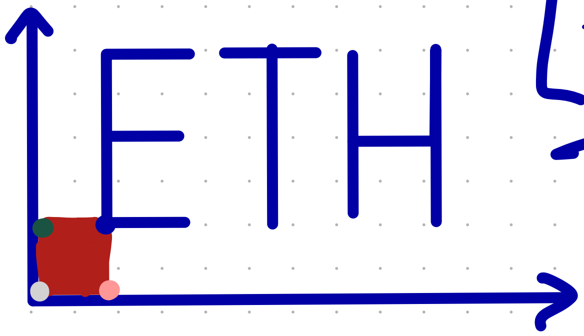
$$\begin{bmatrix}1&0\\0&2\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

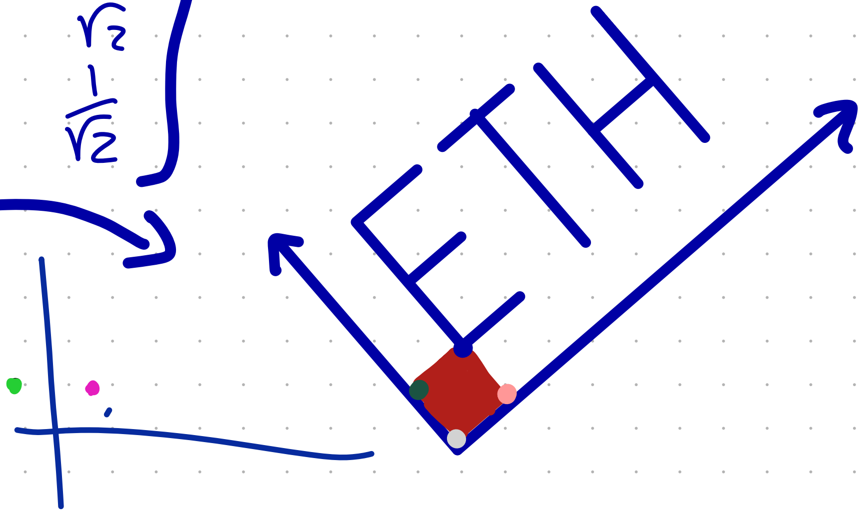
$$\begin{bmatrix}0\\1\end{bmatrix} \longrightarrow \begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

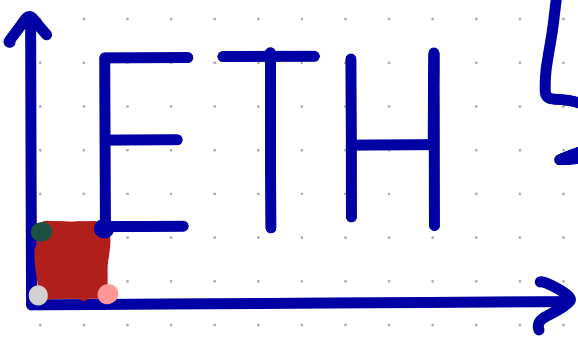


ETH
e_2
e_1
$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
ETH
a_2
a_1

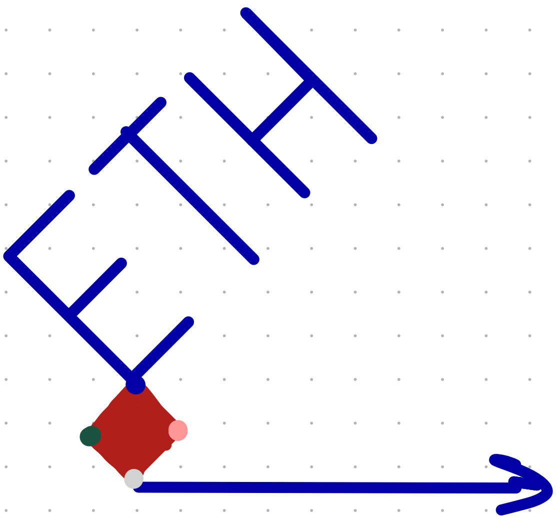


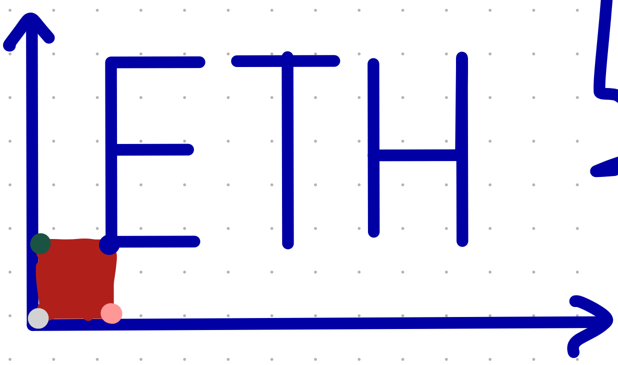
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$



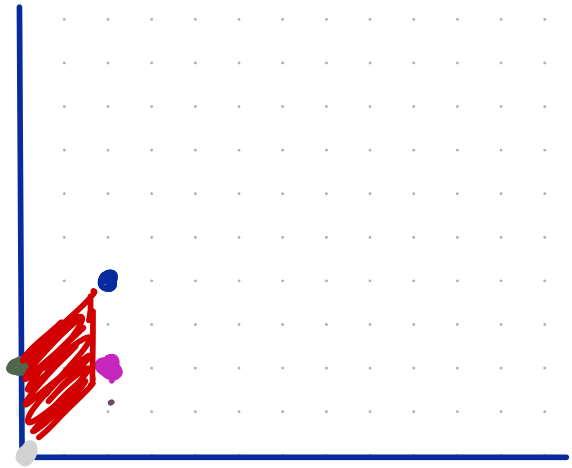


$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

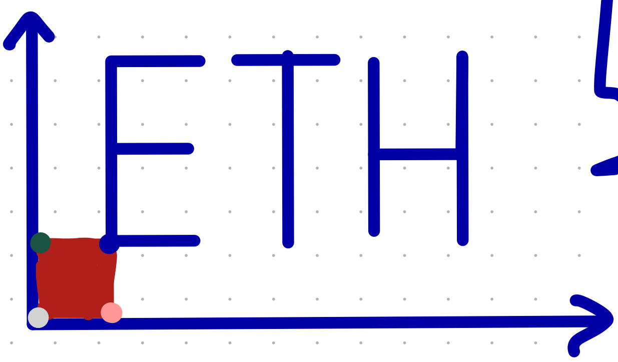




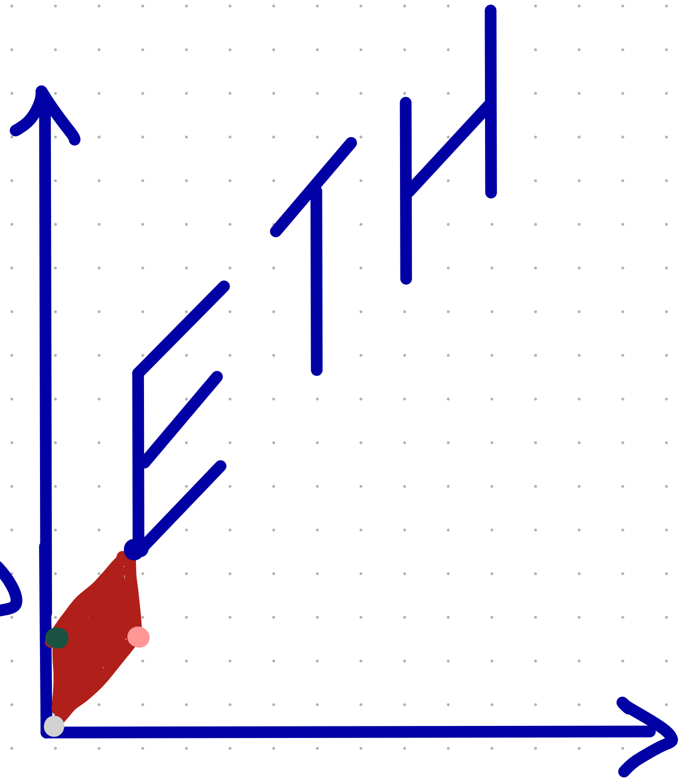
$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

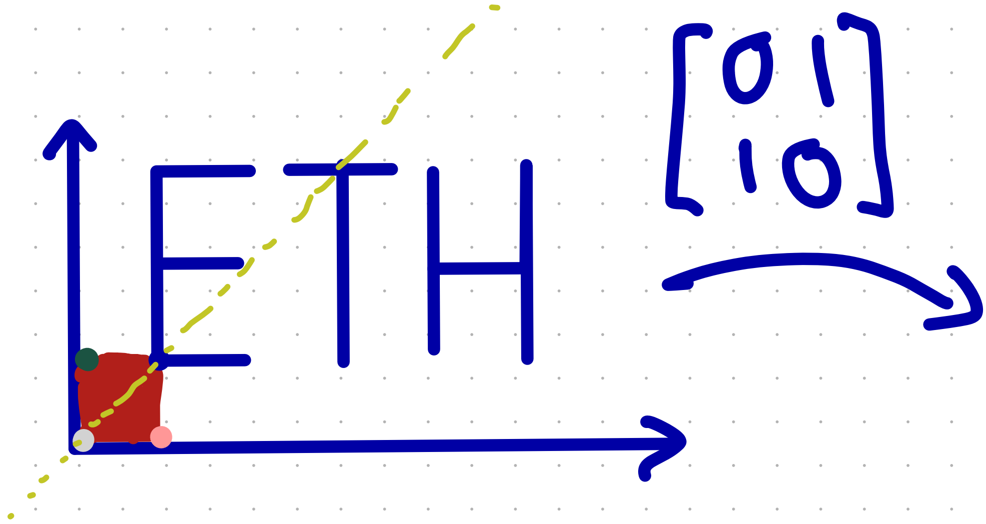


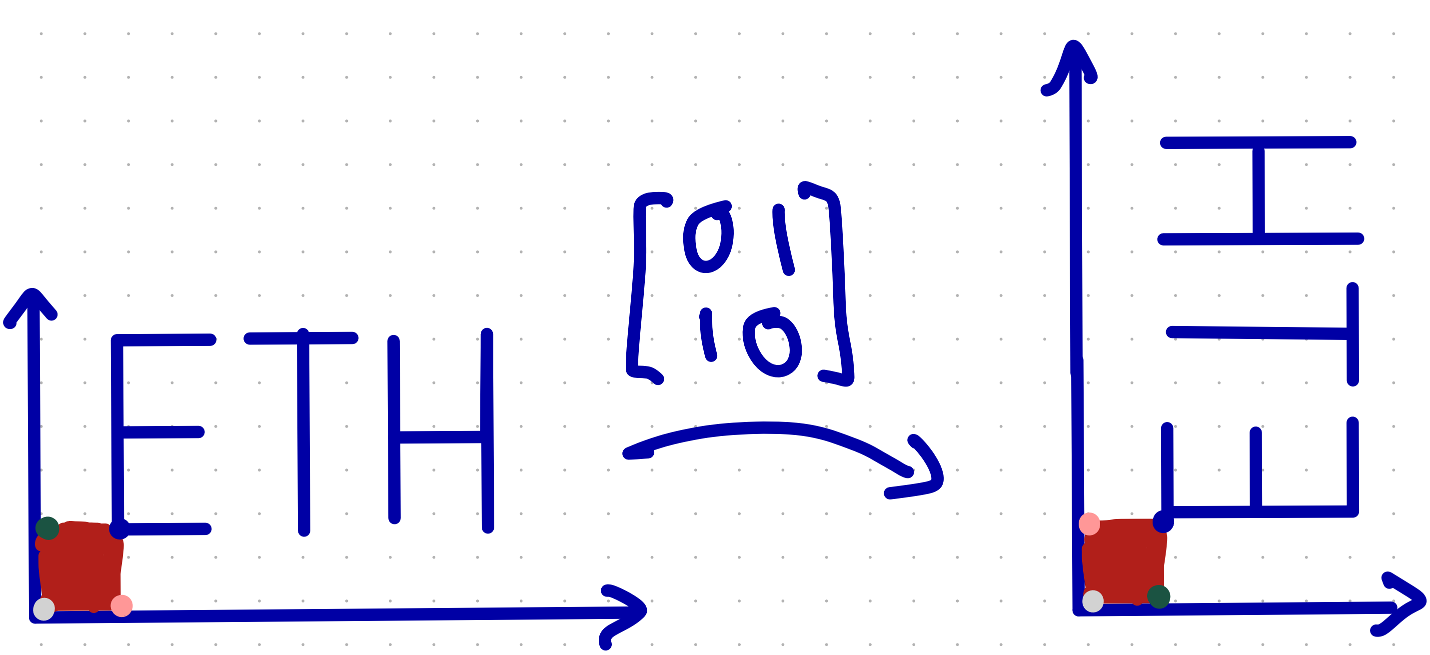
shear



$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$




ETH
0 1
1 0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0 1
1 0

Linear Transformations.

Definition 5.0.2.

Given two vector spaces U and V a linear transformation T is a map

$$T: U \longrightarrow V$$

s.t. $\forall u_1, u_2 \in U$ $\quad T(u_1 + u_2) = T(u_1) + T(u_2)$

$\forall u \in U, \alpha \in \mathbb{R}$ $\quad T(\alpha u) = \alpha T(u)$.

Prop 5.0.6

Let $T: U \to V$ be a linear transformation then $\forall u_1, \ldots, u_n \in U$, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ we have

$$T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n) =$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_n T(u_n).$$

Clicker question

is $x \longmapsto Ax + b$ a L.T. ?

$$A(x_1 + x_2) + b = Ax_1 + Ax_2 + b$$

$$Ax_1 + b + Ax_2 + b = Ax_1 + Ax_2 + 2b$$

$$A\alpha x + b \neq \alpha(Ax + b)$$

<u>Prop:</u> Given two Linear Transformations L, T from U to V and a basis $u_1, \dots, u_n$ of U if $L(u_k) = T(u_k)$ $\forall_k$ then $L = T$.

<u>Proof:</u> For any $u \in U$, $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$

$$
\begin{aligned}
T(u) = T(\alpha_1 u_1 + \cdots + \alpha_n u_n) &= \alpha_1 T(u_1) + \cdots + \alpha_n T(u_n) \\
&= \alpha_1 L(u_1) + \cdots + \alpha_n L(u_n) \\
&= L(u).
\end{aligned}
$$