Linear Algebra

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* Please take a look at the typed lecture notes

Last lecture:

Prop: Given two Linear Transformations L, T from U to V and a basis $u_1, \dots, u_n$ of U if $L(u_k) = T(u_k)$ $\forall_k$ then $L = T$.

Proof: For any $u \in U$, $u = \alpha_1 u_1 + \dots + \alpha_n u_n$

$$
\begin{aligned}
T(u) = T(\alpha_1 u_1 + \dots + \alpha_n u_n) &= \alpha_1 T(u_1) + \dots + \alpha_n T(u_n) \\
&= \alpha_1 L(u_1) + \dots + \alpha_n L(u_n) \\
&= L(u).
\end{aligned}
$$

Prop: Given U and V vector spaces and $u_1, \dots, u_n \in U$ a basis of U for any $v_1, \dots, v_n \in V$ there exists a L.T. T s.t. $T(u_k) = v_k$.

Proof: Given $x \in U$, $x = \alpha_1 u_1 + \dots + \alpha_n u_n$ and define T as $T(x) = \alpha_1 T(u_1) + \dots + \alpha_n T(u_n)$

(HW: show that T is a L.T.)

Examples

(A) $T: \mathbb{R}^n \to \mathbb{R}^n$ L.T.

$T(x) = x$

(B) $T: \mathbb{R}^n \longrightarrow \mathbb{R}$

$T(x) = \|x\|$ not L.T.

(C) $T: \mathbb{R}^n \longrightarrow \mathbb{R}$

$T(x) = v^T x$ L.T.

(for some fixed v)

(D) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ not L.T.

$T(x) \longrightarrow \dfrac{x}{\|x\|}$

(E) $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$T(x) = Ax$ L.T

(for A some fixed $m \times n$ matrix)

$A(x+y) = Ax + Ay$

$A(\alpha x) = \alpha A x.$

Prop: (5.0.10): For any L.T. $T: \mathbb{R}^n \to \mathbb{R}^m$ then exists a matrix $A \in \mathbb{R}^{m \times n}$ s.t. $T(x) = Ax$.

Proof: Let $e_1, \dots, e_n$ be the canonical basis in $\mathbb{R}^n$

$(e_i)_j = \delta_{ij}$.

$x \in \mathbb{R}^n$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$$T(x) = T(x_1 e_1 + \cdots + x_n e_n) = x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}}_{m \times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = Ax$$

A is the matrix with columns $T(e_1) \dots T(e_n)$. $\square$.

Prop: (5.0.11) $T: \mathbb{R}^n \to \mathbb{R}^m$, $L: \mathbb{R}^m \to \mathbb{R}^p$

matrix A $(m \times n)$ $\quad$ matrix B $(p \times m)$ $\quad$ $T(x) = Ax$, $L(x) = Bx$

$$L \circ T : \mathbb{R}^n \longrightarrow \mathbb{R}^p$$

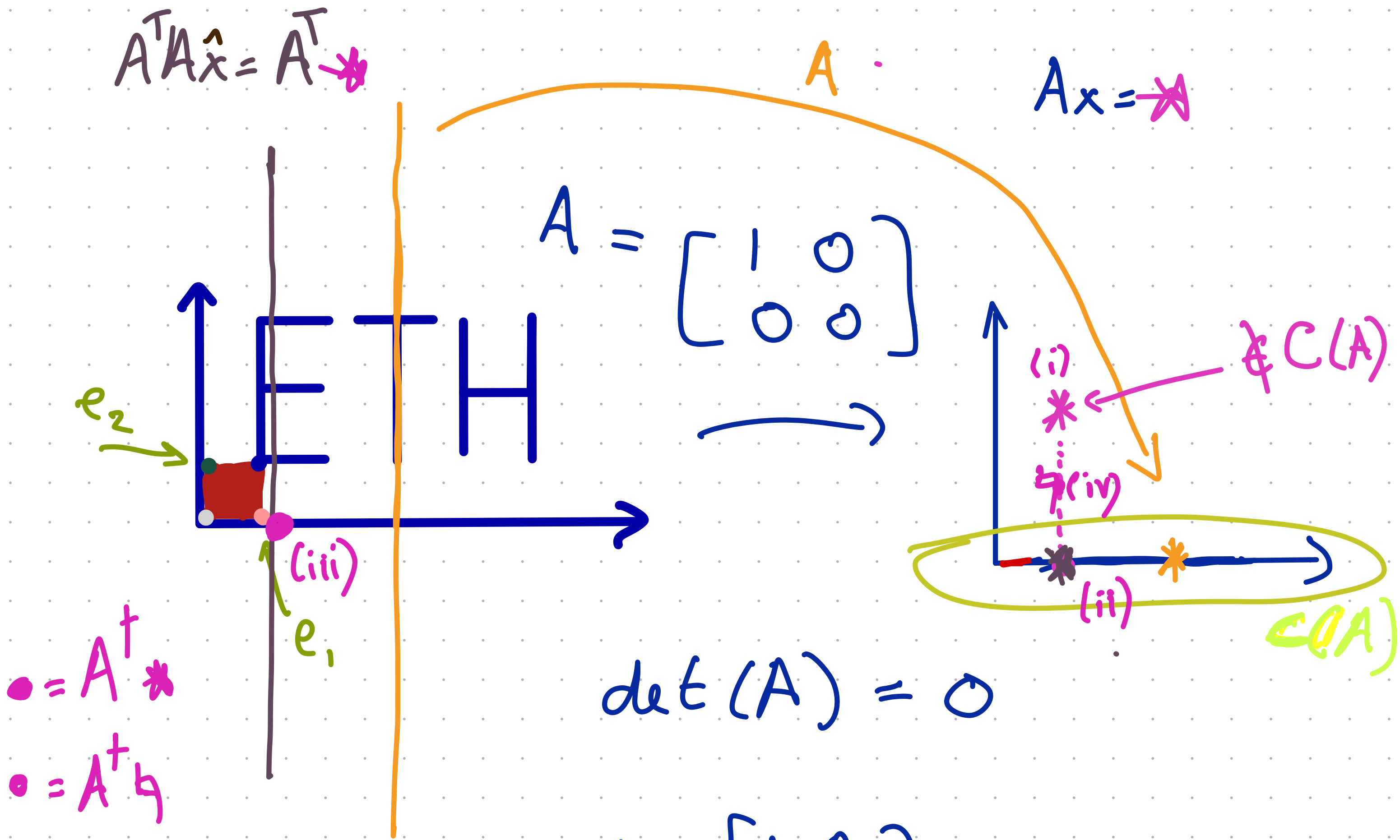
$$L \circ T(x) = L(T(x))$$

$$L \circ T(x) = B(Ax) = BAx.$$

Determinants

Determinant of a square matrix A $(n\times n)$ is the (signed) volume of the image of the unit cube by A.

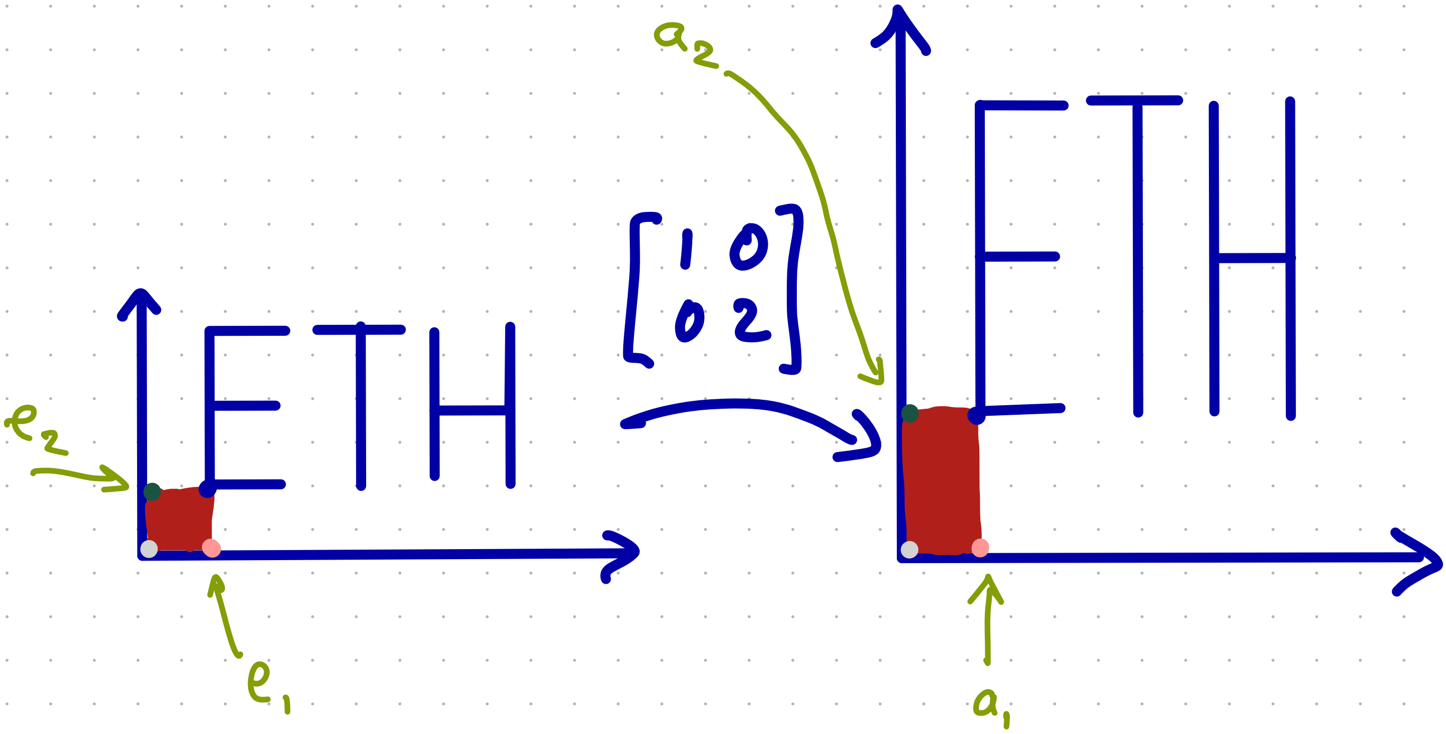
It measures how much "the space is expanded" by applying A.

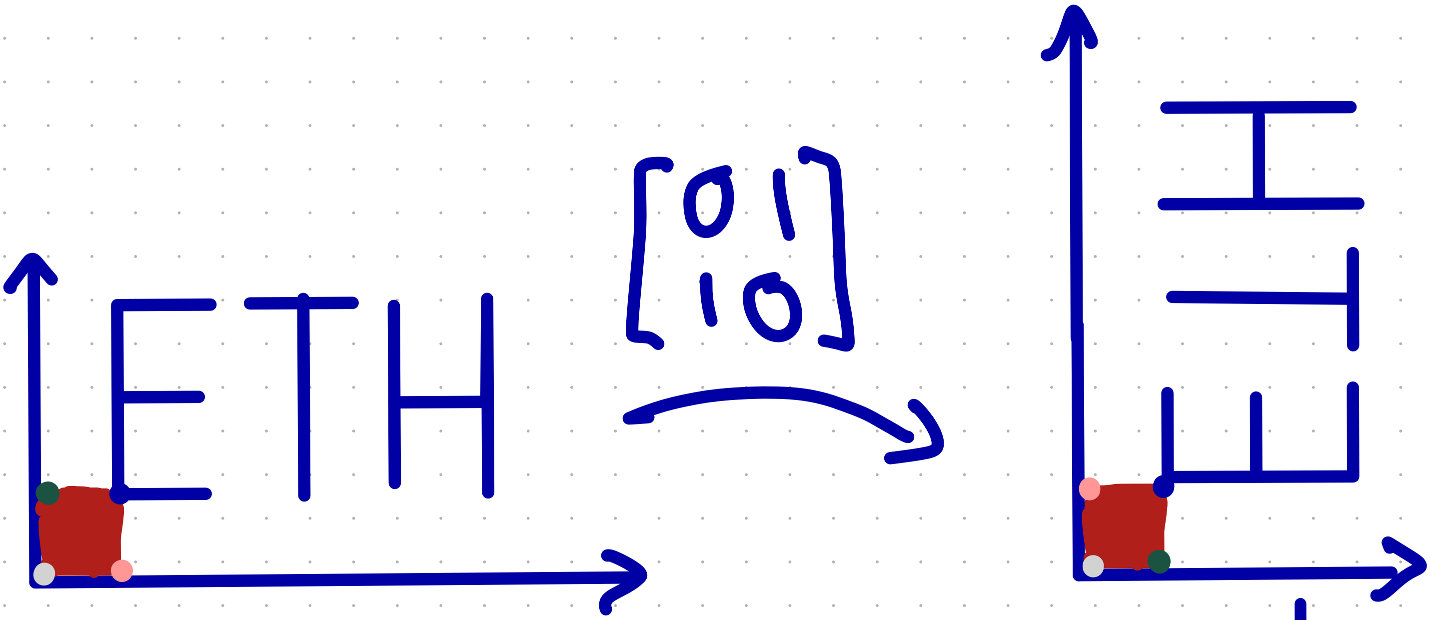


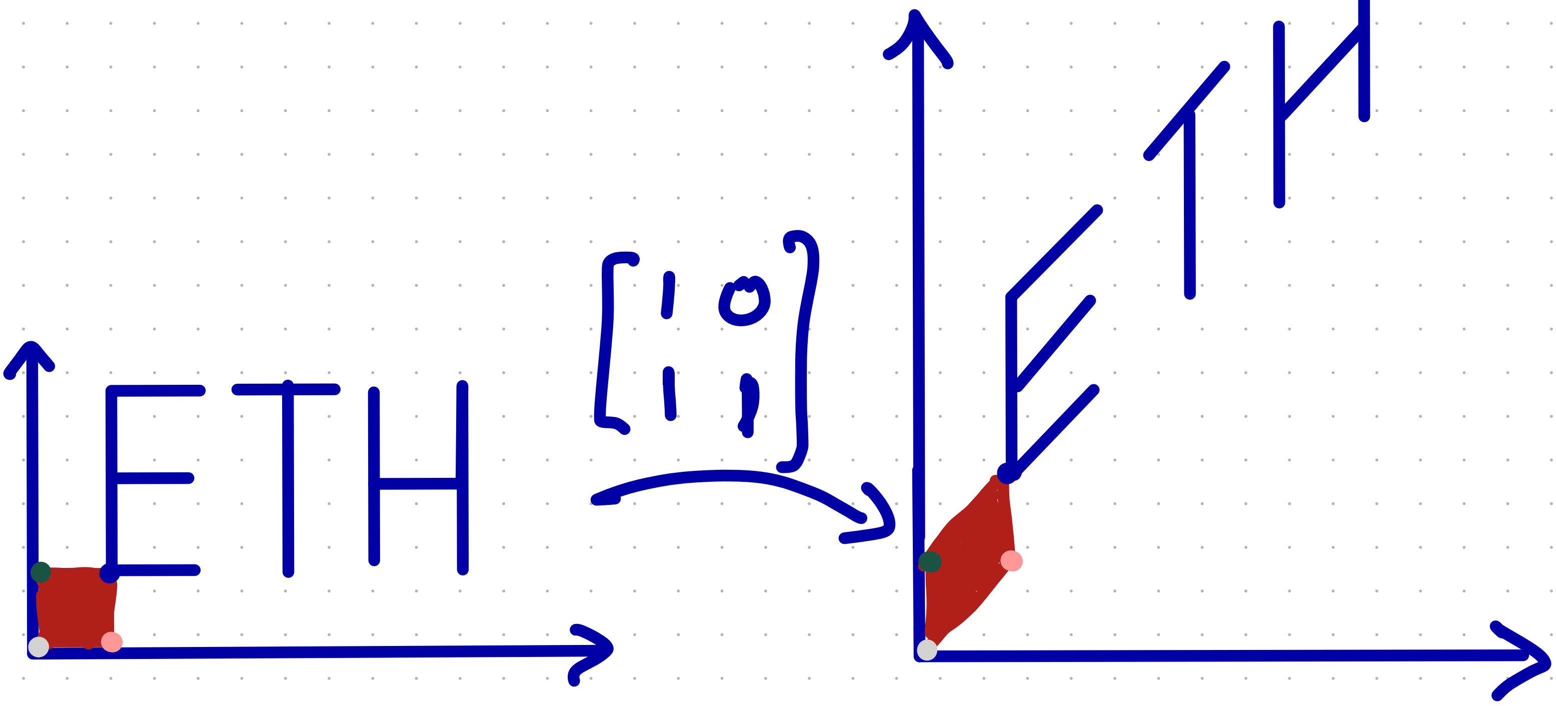
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\det(A) = 0$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \det(A) = -1.$$


ETH
e_2
e_1
$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$
a_2
a_1
ETH


ETH
$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$


ETH
$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

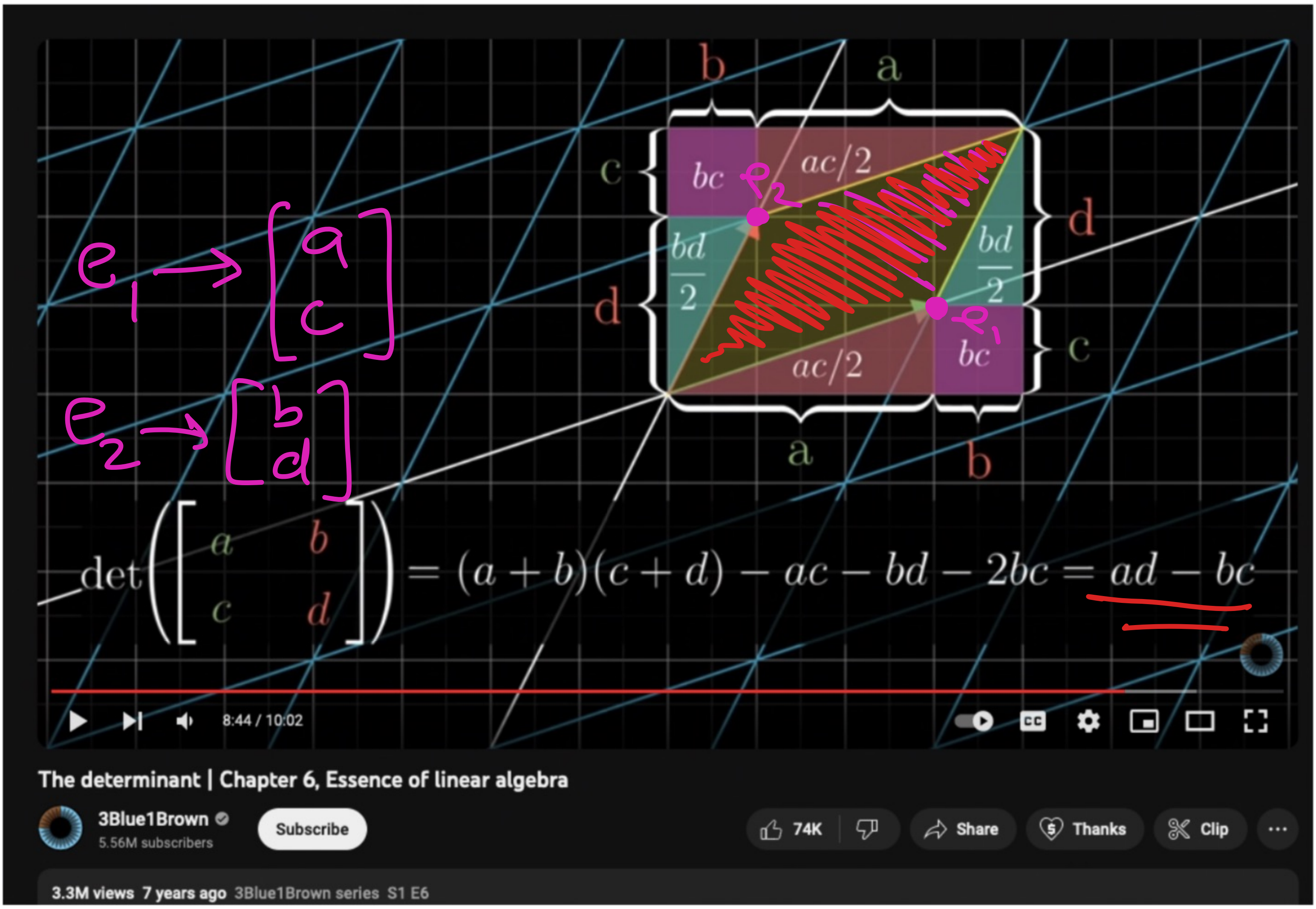


FIGURE 4. Calculation in 3Blue1Brown's video (see Remark 5.1.1) computing the determinant of a 2×2 matrix as the area of the image of the unit square after a linear transformation (that does not change orientation).

Theorem 5.1.2: A matrix $A \in \mathbb{R}^{n \times n}$ is invertible iff (if and only if) $\det(A) \neq 0$.

— $\Rightarrow$ — 2×2 matrices —

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

— $\Leftarrow$ —

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a or b is non-zero
c or d is non-zero.

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \alpha_1 \begin{bmatrix} -b \\ a \end{bmatrix}$$

$$\begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} w \\ y \end{bmatrix} = \alpha_2 \begin{bmatrix} d \\ -c \end{bmatrix}$$

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} = 1 \iff \begin{bmatrix} a & b \end{bmatrix}\alpha_2\begin{bmatrix} d \\ -c \end{bmatrix} = 1$$

$$\alpha_2(ad-bc) = 1$$

$$\alpha_2 = \frac{1}{ad-bc} = \frac{1}{\det(A)}$$

$$\begin{bmatrix} c & d \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} = 1$$

$$\iff \begin{bmatrix} c & d \end{bmatrix}\alpha_1\begin{bmatrix} -b \\ a \end{bmatrix} = 1$$

$$\alpha_1(-cb+da) = 1$$

$$\alpha_1 = \frac{1}{\det(A)}.$$

$$\begin{bmatrix} w \\ y \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d \\ -c \end{bmatrix}$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} -b \\ a \end{bmatrix}$$

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(A^{-1}) \; ?$$

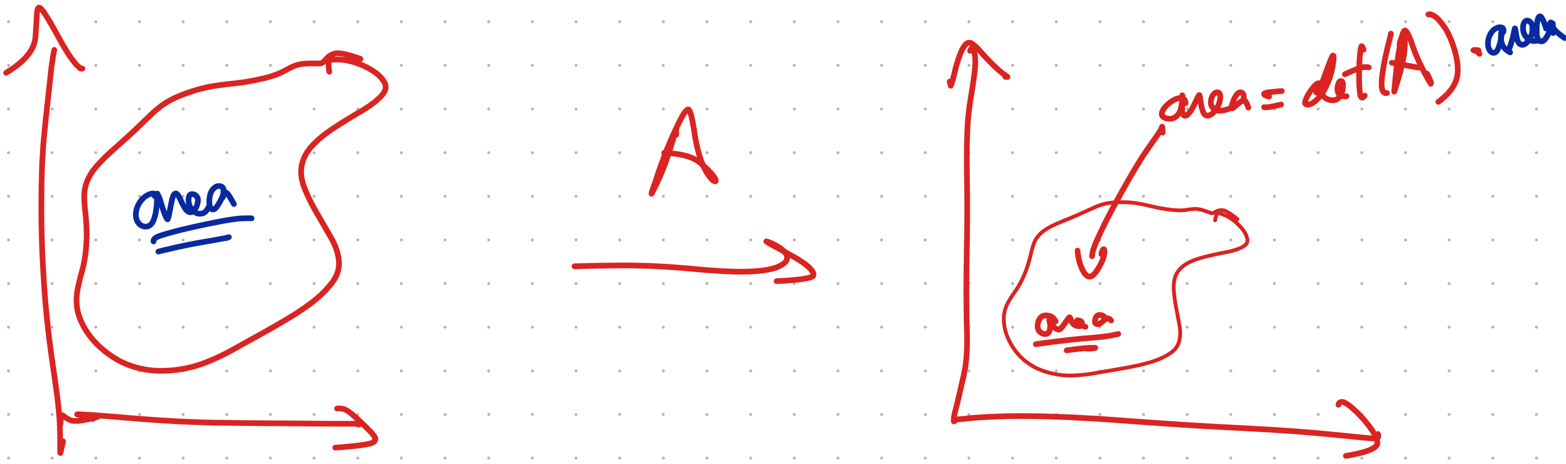
$$\det(A^{-1}) = \begin{vmatrix} \frac{d}{\det(A)} & \frac{-b}{\det(A)} \\ \frac{-c}{\det(A)} & \frac{a}{\det(A)} \end{vmatrix} = \frac{1}{\det(A)^2}(da-bc)$$

$$= \frac{1}{\det(A)}.$$

Theorem: 5.1.4. $A, B \in \mathbb{R}^{n\times n}$

$$\det(AB) = \det(A)\det(B).$$

$$ABx = A(Bx)$$



Def of Determinant for $n\times n$ matrices

Def: 5.1.5. (Sign of Permutation)

Given a permutation
$$\sigma : \{1, \dots, n\} \to \{1, \dots, n\}.$$

$\mathrm{sign}(\sigma) = \mathrm{sgn}(\sigma) =$

$$= \begin{cases} 1 & \text{if } \left|\{(i,j) \in \{1,\dots,n\} \times \{1,\dots,n\} \text{ s.t. } i<j, \sigma(i)>\sigma(j)\}\right| \text{ is even} \\ -1 & \text{if } \quad '' \qquad '' \qquad \text{is odd.} \end{cases}$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \to \begin{pmatrix}1\\2\\3\end{pmatrix} \quad +1$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \to \begin{pmatrix}2\\1\\3\end{pmatrix} \quad -1$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \to \begin{pmatrix}3\\2\\1\end{pmatrix} \quad -1$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \to \begin{pmatrix}2\\3\\1\end{pmatrix} \quad +1$$

Exp. Challenge:

- exactly half of permutations have sign 1.
- $\text{sign}(\sigma \circ \gamma) = \text{sign}(\sigma) \cdot \text{sign}(\gamma)$.

Def (Determinant) Given an $n \times n$ matrix A, with entries A_{ij} the determinant is given by

$$\det(A) = \sum_{\sigma \in \Pi_n} \text{sign}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

all permutations of n elements

Prop 5.1.7. P a permutation matrix (corresponding to permutation σ)

$$\det(P) = \operatorname{sign}(\sigma).$$

(HW: Convince yourself of this)

HW:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = \ ?$$