Linear Algebra

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* Please take a look at the typed lecture notes

Last lecture

Def (Determinant) Given an $n \times n$ matrix A, with entries A_{ij} the determinant is given by

$$\det(A) = \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

all permutations of n elements

$n = 1$

$$\det [A_{11}] = A_{11}$$

$n = 2$ Π_2 permutations of 2 elements

$1 \to 1$	$1 \to 2$
$2 \to 2$	$2 \to 1$
$\operatorname{sgn} = 1$	$\operatorname{sgn} = -1$

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = (+1) A_{11} A_{22} + (-1) A_{12} A_{21}$$

$$= A_{11} A_{22} - A_{12} A_{21}$$

$\underline{n=3}$ # permutations of 3 elements $=3!=6$

$1\to1$	$1\to1$	$1\to2$	$1\to2$	$1\to3$	$1\to3$
$2\to2$	$2\to3$	$2\to1$	$2\to3$	$2\to1$	$2\to2$
$3\to3$	$3\to2$	$3\to3$	$3\to1$	$3\to2$	$3\to1$
sgn 1	sgn -1	sgn -1	sgn 1	sgn 1	sgn -1

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = (+1)A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$

$$\begin{vmatrix} \cdots \end{vmatrix} = \underbrace{\begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & A_{32} & 0 \end{vmatrix}} + \underbrace{\begin{vmatrix} & A_{12} & \\ A_{21} & & \\ & & A_{33} \end{vmatrix}}$$

$$+ \underbrace{\begin{vmatrix} & A_{12} & \\ & & A_{23} \\ A_{31} & & \end{vmatrix}} + \underbrace{\begin{vmatrix} & & A_{13} \\ A_{21} & & \\ & A_{32} & \end{vmatrix} + \begin{vmatrix} & & A_{13} \\ & A_{22} & \\ A_{31} & & \end{vmatrix}}$$

$$\|$$

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \underbrace{A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}} - \underbrace{A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix}} + \underbrace{A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}}$$

Def 5.1.12: Given $A \in \mathbb{R}^{n\times n}$ for each $1 \le i,j \le n$ let $\mathcal{A}_{ij}$ denote the $(n-1)\times(n-1)$ given by removing row i and column j of A.

The co-factor of A are

$$\mathbb{R} \ni C_{ij} = (-1)^{i+j} \det(\mathcal{A}_{ij})$$

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Prop 5.1.13 Let $A \in \mathbb{R}^{n\times n}$ for any $1 \le i \le n$

$$\det(A) = \sum_{j=1}^{n} \underset{\in\mathbb{R}}{A_{ij}}\, \underset{\in\mathbb{R}}{C_{ij}}$$

$$\begin{vmatrix} 5 & 9 & -1 \\ 0 & 1 & 0 \\ 1 & 9 & 1 \end{vmatrix} \overset{\text{Prop 5.1.13 } i=2}{=} 0\,C_{21} + 1\,C_{22} + 0\,C_{23}$$

$$= C_{22} = 1 \cdot \begin{vmatrix} 5 & -1 \\ 1 & 1 \end{vmatrix} = 6.$$

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Prop 5.1.8. Given a triangular (upper or lower) matrix $T \in \mathbb{R}^{n\times n}$ then

$$\det(T) = \prod_{k=1}^{n} T_{kk}$$

in particular $\det(I) = 1$.

$$\begin{vmatrix} a & ? & ? \\ 0 & b & ? \\ 0 & 0 & c \end{vmatrix} = abc$$

$$\begin{vmatrix} a & 0 & 0 \\ ? & b & 0 \\ ? & ? & c \end{vmatrix} = abc$$

Prop 5.1.9. $\det(A) = \det(A^T)$

(HW: Try in a 3×3 example)

Prop 5.1.10. If $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix then $\det Q = 1$ or $\det Q = -1$.

Proof: $1 = \det(I) = \det(QQ^T) = \det(Q)\cdot\det(Q^T) = \det(Q)^2$

so $\det(Q) = 1$ or $\det(Q) = -1$.

Prop 5.1.11. If $A \in \mathbb{R}^{n\times n}$ $\det(A) \neq 0$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof: $1 = \det(I) = \det(AA^{-1}) = \det(A)\cdot\det(A^{-1})$.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Prop 5.1.14.

$A \in \mathbb{R}^{n \times n}$ $\det(A) \neq 0$

$$A^{-1} = \frac{1}{\det(A)} C^T \quad \text{where}$$

C is the matrix of co-factors of A

C_{ij} is the i,j co-factor of A

HW: 2x2 inverse formula we derived is a special case of this.

<u>Cramer's Rule</u>: $A \in \mathbb{R}^{n \times n}$ $\det(A) \neq 0$.

<u>$n = 3$</u> goal solve $Ax = b$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}
= \overset{B_1 :=}{\begin{bmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{bmatrix}}
$$

$$
\det(A) \cdot \underbrace{\det\left(\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix}\right)}_{= x_1} = \det\left(\quad \right)
$$

$$
x_1 = \frac{\det(B_1)}{\det(A)} \qquad B_1 = \begin{bmatrix} b_1 & A_{12} & \vdots \\ b_2 & \vdots & \vdots \\ b_3 & \vdots & \vdots \end{bmatrix}
$$

$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
\begin{bmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{bmatrix}
= \overset{B_2 :=}{\begin{bmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{bmatrix}}
$$

$$
\det(A) \cdot \underbrace{\det\left(\quad \right)}_{x_2} = \det(B_2)
$$

Prop 5.1.16 (Cramer's rule)

Let $A \in \mathbb{R}^{n \times n}$ such that $\det(A) \neq 0$ and $b \in \mathbb{R}^n$, the solution x of $Ax = b$ is given by

$$x_j = \frac{\det(B_j)}{\det(A)}$$

B_j is the matrix obtained by replacing the j-th column of A by b.

Determinants and elimination.

$$PA = LU$$

U: upper triangular

P: permutation

L: lower triangular with 1's in diagonal

$$\det(PA) = \det(LU)$$

$$\underbrace{\det(P)}_{=\operatorname{sgn}(P)}\det(A) = \underbrace{\det(L)}_{=1}\underbrace{\det(U)}_{\text{product of diag elements of } U.}$$

it's 1 or -1

$$\det(A) = \operatorname{sgn}(P)\cdot\underbrace{\det(U)}_{\text{product of diagonal elements of } U.}$$

- every permutation can be written as a composition of swaps of two elements (transposition) $\operatorname{sgn}(P) = (-1)^{\#\text{swaps}}$.

Prop 5.1.18. If $A \in \mathbb{R}^{n\times n}$ and P is a permutation that swaps two elements

$$\det(PA) = -\det(A).$$

Prop 5.1.19. $a_0, \dots, a_n \in \mathbb{R}^n$.

$$\begin{vmatrix} - \alpha_0 a_0^T + \alpha_1 a_1^T - \\ - a_2^T - \\ - a_3^T - \\ \vdots \\ - a_n^T - \end{vmatrix} = \alpha_0 \begin{vmatrix} - a_0^T - \\ - a_2^T - \\ \vdots \end{vmatrix} + \alpha_1 \begin{vmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \end{vmatrix}$$

$$\begin{vmatrix} | & | & | & \\ \alpha_0 a_0 + \alpha_1 a_1 & a_2 & a_3 & \cdots \\ | & | & | & \end{vmatrix} = \alpha_0 \begin{vmatrix} | & | & \\ a_0 & a_2 & \cdots \\ | & | & \end{vmatrix} + \alpha_1 \begin{vmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{vmatrix}$$

Q:

$A \in \mathbb{R}^{n \times n}$

$$\det(\alpha A) = \alpha^n \det(A).$$

Eigenvalues and eigenvectors.

scalar λ vectors $v \neq 0$.

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$\underbrace{(A - \lambda I)}_{\text{is not invertible.}} v = 0$$

$$\det(A - \lambda I) = 0.$$

2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

λ eigenvalue will satisfy

$$\det(A - \lambda I) = 0$$

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc.$$

$$0 = \lambda^2 - (a+d)\lambda + (ad - bc).$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 1$$

$$\lambda^2 + 1 = 0.$$

$$\lambda^2 = -1.$$

Solutions only over Complex Numbers

$$i^2 = -1.$$