Linear Algebra

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* Please take a look at the typed lecture notes

Complex Numbers

$1, 2, 3, 4.$ $\mathbb{N}$

$x + 10 = 5$ needs negative numbers to have solutions $\mathbb{Z}$

$10x = 5$ we need rational numbers $\mathbb{Q}$

$\frac{a}{b}$, $a, b \in \mathbb{Z}$.

$x^2 = 2$ has no solution over $\mathbb{Q}$.

we need $\mathbb{R}$.

$x^2 = -1.$ we need $\mathbb{C}$.

$i^2 = -1.$ $\left(i = \sqrt{-1}\right)$

$$\mathbb{C} = \{ a + ib \ : \ a, b \in \mathbb{R} \}$$

$$(a+ib) + (x+iy) = (a+x) + i(b+y)$$

$$(a+ib)(x+iy) = ax + ibx + iay + i^2 by$$
$$= (ax - by) + i(bx + ay).$$

$$(a+ib)(a-ib) = a^2 - i^2 b^2 = a^2 + b^2.$$

$$\frac{a+ib}{x+iy} = \frac{(x-iy)(a+ib)}{(x-iy)(x+iy)} = \frac{(ax+by)+i(xb-ya)}{x^2+y^2}.$$

$z \in \mathbb{C}$ $\quad z = a+ib$

$$\mathcal{R}(z) := a$$

$$\mathcal{I}(z) := b$$

$$|z| := \sqrt{a^2+b^2}$$

$$\overline{z} := a - ib$$

$z \in \mathbb{C}$ $\quad z_1, z_2 \in \mathbb{C}$

$$|z|^2 = \overline{z}\, z$$

$$z_1 z_2 = z_2 z_1$$

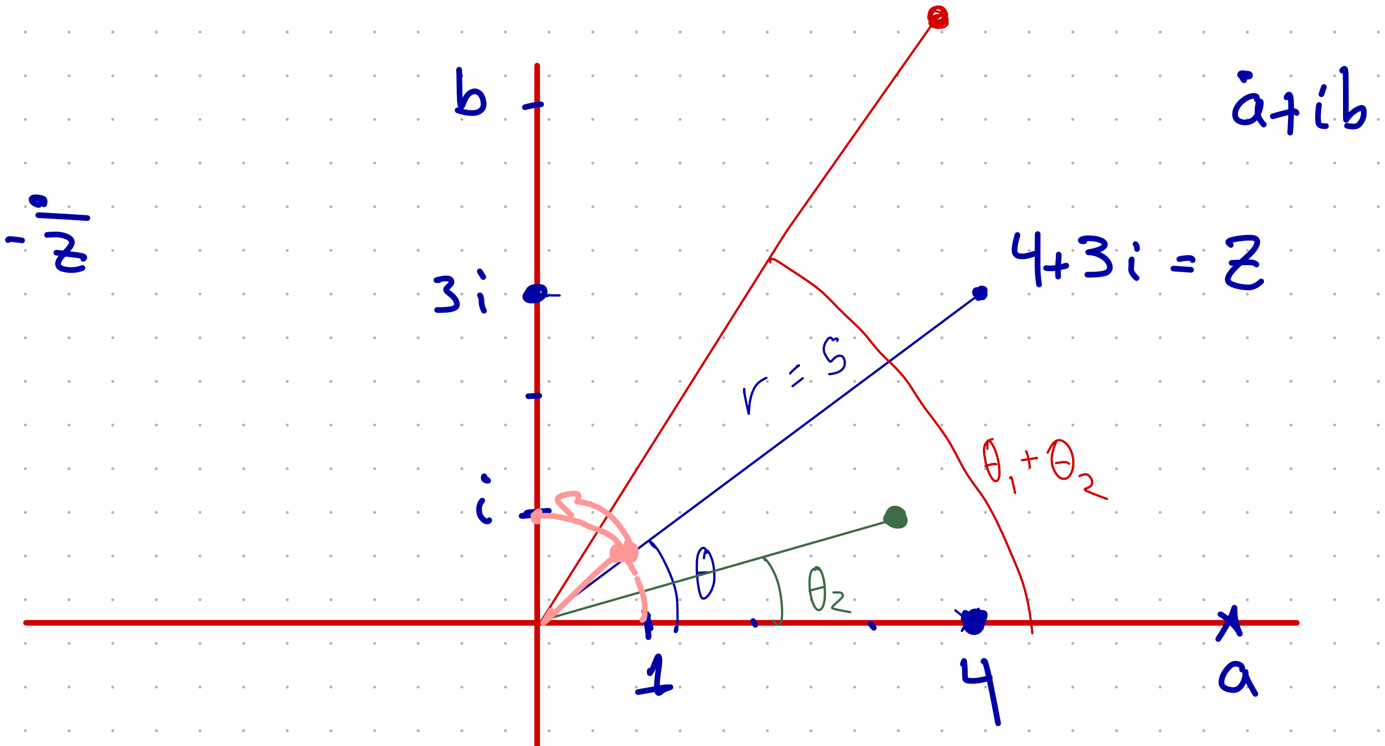
$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{|z_2|^2}.$$

Fact 6.0.1. (Euler's formula) $\quad \theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

If $\theta = \pi$ $\quad e^{i\pi} = -1$ $\quad \boxed{e^{i\pi} + 1 = 0}$



$-z$

$\bar{z}$

$4+3i = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

modulus

argument

$r_1e^{i\theta_1}\, r_2e^{i\theta_2}$

$=(r_1r_2)e^{i(\theta_1+\theta_2)}$

$$x^2 = i$$

$$(a+ib)^2 = i$$

$$\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + 2\,\frac{2}{4}\,i - \frac{2}{4} = i.$$

Theorem 6.0.3. (Fundamental Theorem of Algebra)
Any degree n non-constant polynomial $(n \geq 1)$

$$P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$

$\alpha_k \in \mathbb{C}$.
$\alpha_n \neq 0$.

has a zero $\lambda \in \mathbb{C}$. (also called a "root")

$$P(\lambda) = 0.$$

Corollary: $P(z) = \alpha_n z^n + \cdots + \alpha_0 \quad \alpha_n \neq 0 \quad n \geq 1.$

(#) $P(z) = \alpha_n (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$

$\lambda_k \in \mathbb{C}$.

zeros might be repeated. The number of times $\lambda \in \mathbb{C}$ shows up in (#) is called the algebraic multiplicity of the zero.

Complex Valued matrices and vectors.

$$v \in \mathbb{C}^n \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad v_k \in \mathbb{C}.$$

$$A \in \mathbb{C}^{m \times n}. \qquad A^* = \overline{A^T}$$

$$\|v\|^2 = \sum_{i=1}^{n} |v_i|^2 = \sum_{i=1}^{n} v_i \overline{v_i} = v^* v$$

$n=2$

$v^* v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^* \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = [\bar{v}_1 \bar{v}_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

a subspace $U \subseteq \mathbb{C}^n$ is defined the same way. but with scalar $\alpha \in \mathbb{C}$.

if $v \in U$ then $\alpha v \in U$ also for $\alpha \in \mathbb{C}$.

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$v_1, \dots, v_k \in \mathbb{C}^n$ the $\operatorname{Span}(v_1, \dots, v_k) =$

$$= \{\alpha_1 v_1 + \dots + \alpha_k v_k : \alpha_1, \dots, \alpha_k \in \mathbb{C}\}$$

We say $v_1, \dots, v_k \in \mathbb{C}^n$ are linearly ind if

$$\alpha_1 v_1 + \dots + \alpha_k v_k = 0 \Rightarrow \alpha_1, \dots, \alpha_k = 0$$

If $v_1, \dots, v_k \in \mathbb{C}^m$ span U and are linearly ind they are called a basis.

— " —

$$(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) =$$

$$= \underbrace{(\cos\theta\cos\varphi - \sin\theta\sin\varphi)}_{\cos(\theta+\varphi)} + i\underbrace{(\cos\theta\sin\varphi + \sin\theta\cos\varphi)}_{\sin(\theta+\varphi)}.$$

Eigenvalues and eigenvectors.

Guiding Example Fibonacci Numbers.

$$F_0 = 0, \; F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}.$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad g_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

$$g_n = M g_{n-1} \qquad g_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Def: Given $A \in \mathbb{R}^{n \times n}$, we say $\lambda \in \mathbb{C}$ is an eigenvalue of A and $v \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of A, associated with the eigenvalue λ, when

$$Av = \lambda v.$$

We call λ, v an eigenvalue-eigenvector pair

if $\lambda \in \mathbb{R}$ we call it a real eigenvalue.
and λ, v a real eigenvalue-eigenvector pair.

Goal : find eigenvalues of $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$Mv = \lambda v \qquad v \neq 0$$

$$(M - \lambda I)v = 0 \implies M - \lambda I \text{ is not invertible}$$

$$\iff \det(M - \lambda I) = 0$$

$$0 = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - (1)(1) =$$

$$= \lambda^2 - \lambda - 1$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$$\boxed{\lambda_1 = \frac{1+\sqrt{5}}{2}}, \quad \boxed{\lambda_2 = \frac{1-\sqrt{5}}{2}}.$$

Goal: find eigenvectors.

$$Mv_1 = \lambda_1 v_1 \iff (M - \lambda_1 I)v_1 = 0$$

$$v_1 \in N(M - \lambda I)$$

$$\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} (v_1)_1 \\ (v_1)_2 \end{bmatrix} = 0$$

if $(v_1)_2 = 1$ $\quad \left[1 \quad -\frac{1+\sqrt{5}}{2}\right]\begin{bmatrix}(v_1)_1 \\ 1\end{bmatrix} = 0$

$$(v_1)_1 = \frac{1+\sqrt{5}}{2}$$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\quad v_1 = \begin{bmatrix}\frac{1+\sqrt{5}}{2} \\ 1\end{bmatrix}$ $\quad$ (HW: check)

$\lambda_2 = \frac{1-\sqrt{5}}{2}$, $\quad v_2 = \begin{bmatrix}\frac{1-\sqrt{5}}{2} \\ 1\end{bmatrix}$

$A \in \mathbb{R}^{n\times n}$ $\quad \lambda, v$ are evalue-evector pair of A

$$Av = \lambda v \qquad v \neq 0.$$

$$(A-\lambda I)v = 0$$

$\det(A-\lambda I) = 0$ $\leftarrow$ can find evalue

$v \in N(A-\lambda I)$. $\leftarrow$ can find evector.

Prop 6.1.2. Let $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$ is a (real) eigenvalue of A if and only if $\det(A-\lambda I)=0$ and v is an associated eigenvector iff

$v \in N(A-\lambda I) \setminus \{0\}$.

Prop 6.1.3. $\det(A-\lambda I)$ is a polynomial in λ of degree n. The coefficient of the term λ^n is equal to: $(-1)^n$. (try a 3×3 example)

Theorem 6.1.4. Every matrix $A \in \mathbb{R}^{n \times n}$ has an eigenvalue (perhaps $\lambda \in \mathbb{C}$).

Proof: Fundamental Thm of Algebra.

We'll mostly focus on real evalues but the theory adapts naturally to $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n$. $v \in N(A-\lambda I) \setminus \{0\}$ viewed as a subspace of $\mathbb{C}^n$.

back to Fibonacci.

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2}, \quad v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

v_1, v_2 are a basis for $\mathbb{R}^2$.

$$\begin{bmatrix}1\\0\end{bmatrix} = g_0 = \alpha_1 v_1 + \alpha_2 v_2.$$

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = \frac{-1}{\sqrt{5}}.$$

$$g_n = A g_{n-1} = A^2 g_{n-2} = A^n g_0$$

$$g_n = A^n(\alpha_1 v_1 + \alpha_2 v_2)$$

$$g_n = A^n \alpha_1 v_1 + A^n \alpha_2 v_2$$

$$= \alpha_1 \underbrace{A^n v_1} + \alpha_2 A^n v_2.$$

$$A v_1 = \lambda_1 v_1$$

$$A^2 v_1 = A(\lambda_1 v_1) = \lambda_1 A v_1 = \lambda_1^2 v_1$$

$$A^n v_1 = \lambda_1^n v_1$$ (Prove by ind)

$$g_n = \alpha_1 \lambda_1^n v_1 + \alpha_2 \lambda_2^n v_2$$

$$g_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n v_1 + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n v_2$$

$$g_n = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} \qquad v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$