Linear Algebra

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* Please take a look at the typed lecture notes

Last Lecture:

Fibonacci Numbers.

$F_0 = 0, F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$.

$$g_n = \alpha_1 \lambda_1^n v_1 + \alpha_2 \lambda_2^n v_2$$

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Prop 6.1.6: If λ and v are an evalue evector pair of A, then for any $k \geq 1$ (integer)

λ^k, v are an evalue/evector pair for A^k.

Proof: Induction.

base case: $k=1$ trivial

if it holds for $k-1$ then

$$A^k v = A(A^{k-1}v) =$$
$$= A(\lambda^{k-1}v) =$$
$$= \lambda^{k-1}Av = \lambda^{k-1}\lambda v$$
$$= \lambda^k v.$$

Prop 6.1.1006. Let A be an invertible matrix. If λ, v are an e-value/e-vector pair for A then λ^{-1}, v are an e-value/e-vector pair for A^{-1}.

Proof: $Av = \lambda v$, $\lambda \neq 0$. $(v \neq 0)$

$$A^{-1}Av = A^{-1}\lambda v \Leftrightarrow v = A^{-1}\lambda v \Leftrightarrow$$

$\Leftrightarrow \quad \frac{1}{\lambda} v = A^{-1} v$

$\square$

Prop 6.1.7. Let $A \in \mathbb{R}^{n \times n}$ and let $v_1, \dots, v_k \in \mathbb{R}^n$ be eigenvectors corresponding to eigenvalues $\lambda_1, \dots, \lambda_k$.

If $\lambda_1, \dots, \lambda_k$ are all distinct then $v_1, \dots, v_k$ are linearly independent.

<u>Proof:</u> We will prove this by contradiction.

Assume that $v_1, \dots, v_k$ are linearly dependent.

Let d_i denote the dimension of span $v_1, \dots, v_i$

Let j be the smallest i s.t. $d_i < i$.

$$\left| \begin{array}{l} v_1, \dots, v_j \text{ are lin. dep.} \\ v_1, \dots, v_{j-1} \text{ are lin ind.} \end{array} \right.$$

It must be that

$$v_j = \alpha_1 v_1 + \dots + \alpha_{j-1} v_{j-1}$$

$$A v_j = A \alpha_1 v_1 + \dots + A \alpha_{j-1} v_{j-1}$$

$$\underbrace{A v_j}_{\lambda_j v_j} = \alpha_1 \underbrace{A v_1}_{\lambda_1 v_1} + \dots + \alpha_{j-1} A v_{j-1}$$

.....

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$v_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda_j v_j = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{j-1} \lambda_{j-1} v_{j-1}$

$(\alpha_1 v_1 + \alpha_2 v_2 + \cdots)$

$\underbrace{\beta_1 v_1 + \cdots + \beta_{j-1} v_{j-1}} + \beta_j v_j = 0$

$v_j = \frac{-1}{\beta_j} \beta_1 v_1 + \cdots$

$\exists\ \beta_j \neq 0.$

$\lambda_j \alpha_1 v_1 + \lambda_j \alpha_2 v_2 + \cdots + \lambda_j \alpha_{j-1} v_{j-1} = \alpha_1 \lambda_1 v_1 + \cdots + \alpha_{j-1} \lambda_{j-1} v_{j-1}$

$\alpha_1 (\lambda_j - \lambda_1) v_1 + \alpha_2 (\lambda_j - \lambda_2) v_2 + \cdots + \alpha_{j-1} (\lambda_j - \lambda_{j-1}) v_{j-1} = 0$

$0 \neq \lambda_j - \lambda_i$ for all $i < j$ and not all α's are zero

so $v_1, \ldots, v_{j-1}$ are lin dep $\lightning$

Theorem 6.1.8 Let $A \in \mathbb{R}^{n \times n}$ with n distinct (real) eigenvalues. Then there exists a basis of $\mathbb{R}^n$ made of eigenvectors of A.

Prop: if λ is an e-value of M then it is also an e-value of M^T.

Many Properties of eigenvalues follow Corollary 6.0.4

$$(-1)^n \det(A - zI) = \det(zI - A) = \qquad (31)$$
$$(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n).$$

Characteristic Polynomial of a Matrix.

Prop 6.1.9. Given $A \in \mathbb{R}^{n\times n}$ the eigenvalues of A are the same as A^T.

Proof:
$$\det(A - zI) = \det((A - zI)^T)$$
$$= \det(A^T - zI).$$

$((A-zI)^T = A^T + zI)$

so it has the same characteristic polynomial.

Def 6.1.10. Given $A \in \mathbb{R}^{n\times n}$ the trace of A is defined as

$$\mathrm{Tr}(A) := \sum_{i=1}^{n} A_{ii}$$

Prop 6.1.11. Let $A \in \mathbb{R}^{n\times n}$ and $\lambda_1, \dots, \lambda_n$ its n eigenvalues, repeated as in (31).

multiplicity, the:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda_1 = \frac{1+\sqrt{5}}{2}$ $\lambda_1 + \lambda_2 = 1 = \mathrm{tr}$

$\lambda_2 = \frac{1-\sqrt{5}}{2}$ $\lambda_1 \lambda_2 = -1 = \det.$