Linear Algebra

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* Please take a look at the typed lecture notes

Last Lecture:

Def 6.1.10. Given $A \in \mathbb{R}^{n\times n}$ the trace of A is defined as $Tr(A) := \sum_{i=1}^{n} A_{ii}$

Prop 6.1.11. Let $A \in \mathbb{R}^{n\times n}$ and $\lambda_1, \dots, \lambda_n$ its n eigenvalues, repeated as in (31).

of

$(-1)^n \det(A - zI) = \det(zI - A) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$. (31)

Characteristic Polynomial of a Matrix.

multiplicity, then:

$$Tr(A) = \sum_{i=1}^{n} \lambda_i$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

Proof (of Prop 6.1.11)

(Goal 1)

$$\det(A) = \prod_{i=1}^{n} \lambda_i .$$

Let's set $z = 0$ in (31)

$$(-1)^n \det(A - 0I) = \prod_{i=1}^{n} (-\lambda_i)$$

$$(-1)^n \det(A) = (-1)^n \prod_{i=1}^{n} \lambda_i \checkmark$$

(Goal 2)

$$tr(A) = \sum_{i=1}^{n} \lambda_i .$$

$$\det(zI - A) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$$

$$\det(zI - A) = z^n + \underbrace{\quad}_{-\sum_{i=1}^{n} A_{ii} = -Tr(A)} z^{n-1} + \cdots$$

$$\begin{bmatrix} z-A_{11} & -A_{12} & & \\ -A_{21} & z-A_{22} & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix}$$

$$(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n) = z^n + \underbrace{\quad}_{= -\sum_{i=1}^{n} \lambda_i} z^{n-1} + \cdots + \underline{\quad}\, z + \underbrace{\quad}_{(-1)^n \det(A)}$$

$$\Rightarrow \quad -Tr(A) = -\sum_{i=1}^{n} \lambda_i \qquad \square$$

HW: try your favorite 3×3 ehh.

Caution!

(i) Even though the evalues of A and A^T are the same, the evectors are not!

(ii) eigenvalues of $A+B$ are not easy to write from eigenvalues of A and B.

(iii) Same for AB.

(iv) Gaussian elimination does not preserve eigenvalues. The eigenvalues of A are not the diagonal elements of U in $PA = LU$!

Example 6.1.21.

Let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix: $D_{ij} = 0$ if $i \neq j$.

What are the e-values of D? (and e-vectors)

$\lambda_i = D_{ii}$ is an e-value

and $v_i = e_i$

$$\left(e_i := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i \right)$$

$De_i = D_{ii} e_i$.

what about for a triangular matrix?

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda_1 = \dfrac{1+\sqrt{5}}{2}$

$\lambda_2 = \dfrac{1-\sqrt{5}}{2}$.

HW

Example 6.1.14.

The eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$?

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

There is no real eigenvalue.

$$\lambda_1 = i \quad , \quad \lambda_2 = -i$$

$$A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

$$\rightarrow \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} (v_1)_1 \\ (v_1)_2 \end{bmatrix} = 0$$

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} = (-i)\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 \quad \cdots$$

Prop 6.1.15: Let $Q \in \mathbb{R}^{n \times n}$ be an orthogonal matrix.
If $\lambda \in \mathbb{C}$ is an eigenvalue for Q then $|\lambda| = 1$.

Proof: Let $v \in \mathbb{C}^n$ be an eigenvector ($v \neq 0$) associated with λ.

$$Qv = \lambda v$$

Since Q is orthogonal, $\|Qv\|^2 = \|v\|^2$

$$\|v\|^2 = \|Qv\|^2 = \|\lambda v\|^2 = |\lambda|^2 \|v\|^2.$$

$$\Rightarrow |\lambda| = 1.$$

Repeated eigenvalues.

Example 6.1.16.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \end{aligned}$$

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2.$$

$\lambda = 0$ has algebraic multiplicity 2.

$$N(A - 0I) = N(A) \longleftarrow \text{has dimension } 1.$$

$$N(A) = \left\{ \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \text{ for some } \alpha \in \mathbb{R} \right\}$$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A

There is no basis of $\mathbb{R}^2$ made with eigenvectors of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$\longrightarrow$ $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Nilpotent.

"block Jordan form"

Example 6.1.17

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \qquad \begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 0 \end{aligned}$$

$N(A) = \mathbb{R}^2$.

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

There is a basis of $\mathbb{R}^2$ with eigenvectors of

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Def. Given a matrix A and an eigenvalue λ, we call $\dim N(A - \lambda I)$ the geometric multiplicity of λ.

Unfortunately the geometric multiplicity may be smaller than algebraic multiplicity.

Def: Given $A \in \mathbb{R}^{n\times n}$, if we can build a basis of $\mathbb{R}^n$ with evectors of A we say A has a complete set of eigenvectors.

Example 6.1.19.

Let U be a subspace of $\mathbb{R}^n$ and P the projection matrix for U.

$N(P) = U^\perp$ any $w \in U^\perp$ satisfies $Pw = 0w$.

let $k = \dim U$. $n-k = \dim U^\perp$

$w_1, \dots, w_{n-k}$ lin ind s.t. $Pw_j = 0w_j$ $j = 1, \dots, n-k$.

If $u \in U$ then $Pu = u$.

$u_1, \dots, u_k$ lin ind such that $Pu_i = 1 \cdot u_i$ $i = 1, \dots, k$.

$u_1, \dots, u_k$ k ind e-vectors with evalue 1
$w_1, \dots, w_{n-k}$ $n-k$ ind with evalue 0.

P has a complete set of eigenvectors.

There is an orthonormal basis of $\mathbb{R}^n$ made up of e-vectors of P.

$A \in \mathbb{R}^{m \times n}$

$(e_i)_j = \delta_{ij}$.

Change of basis.

$L : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$x \in \mathbb{R}^n \longrightarrow Ax \in \mathbb{R}^m$

$$x = \sum_{j=1}^{n} x_j e_j \longrightarrow L(x) = \sum_{i=1}^{m} (Ax)_i e_i$$

$u_1, \dots, u_n$ be a basis for $\mathbb{R}^n$

$v_1, \dots, v_m$ be a basis for $\mathbb{R}^m$

$$X = \sum_{j=1}^{n} \alpha_j u_j \longrightarrow L(x) = \sum_{i=1}^{m} \beta_i v_i$$

$$\alpha \longrightarrow \beta$$

what is the matrix $B \in \mathbb{R}^{m \times n}$

s.t. $B\alpha = \beta$.

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

$L(x) = Ax$

$$x = \sum \alpha_j u_j = U\alpha \quad , \quad U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \; n \times n$$

$\alpha = U^{-1}x$

$$L(x) = \sum \beta_i v_i = V\beta \quad , \quad V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \; m \times m$$

$\beta = V^{-1}L(x)$

$$L(x) = Ax = A(U\alpha)$$

$V\beta = L(x)$

$$V\beta = AU\alpha$$

$$\beta = V^{-1}AU\alpha$$

$$B = V^{-1}AU.$$

L, written in the basis $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ is described by the matrix $V^{-1}AU$.

where A is the matrix representing L in the canonical basis.

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{linear transf.}$$

$$L\left(\sum_{j=1}^{n} x_j e_j\right) = \sum_{i=1}^{m} (Ax)_i \, e_i \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$L\left(\sum_{j=1}^{n} \alpha_j u_j\right) = \sum_{i=1}^{m} (B\alpha)_i \, v_i \qquad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$B = V^{-1} A U$$