Linear Algebra

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* Please take a look at the typed lecture notes

* The full notes are now available !!

Last Lecture:

L, written in the basis $u_1, \dots, u_n$ and $v_1, \dots, v_m$ is described by the matrix $V^{-1}AU$.
where A is the matrix representing L in the canonical basis.

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{linear transf.}$$

$$L\left(\sum_{j=1}^{n} x_j \cdot e_j\right) = \sum_{i=1}^{m} (Ax)_i \, e_i \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$L\left(\sum_{j=1}^{n} \alpha_j u_j\right) = \sum_{i=1}^{m} (B\alpha)_i \, v_i \qquad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$B = V^{-1} A U$$

This is called the change of basis of a linear transformation.

back to $n \times n$ matrices.

$A \in \mathbb{R}^{n \times n}$ has a complete set of eigenvectors.

$v_1, \ldots, v_n$ eigenvectors of A

basis of $\mathbb{R}^n$.

any vector v can be written as

$$v = \sum \alpha_i v_i$$

$$Av = \sum \alpha_i A v_i = \sum \alpha_i \lambda_i v_i$$

$$A^{10} v = \sum \alpha_i A^{10} v_i = \sum \alpha_i \lambda_i^{10} v_i .$$

Theorem 6.2.1. Let $A \in \mathbb{R}^{n \times n}$ with a complete set of real eigenvectors $v_1, \ldots, v_n$ and associated eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Let $V \in \mathbb{R}^{n \times n}$ be the matrix with columns $v_1, \ldots, v_n$

$$V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

$$A = V \Lambda V^{-1} \qquad (1)$$

where Λ is an $n \times n$ diagonal matrix with $\lambda_1, \dots, \lambda_n$ in the diagonal.

<u>Proof:</u> $v_1, \dots, v_n$ is a basis thus V is invertible.

it is enough to show

$$V^{-1} A V = \Lambda$$

$$\boxed{\begin{aligned} V^{-1}AV &= \Lambda \iff \\ VV^{-1}AV &= V\Lambda \iff \\ VV^{-1}AVV^{-1} &= V\Lambda V^{-1} \iff \\ A &= V\Lambda V^{-1} \end{aligned}}$$

let's see what is the j-th column of $V^{-1}AV$.

$$(V^{-1}AV)_{\cdot j} = V^{-1}AVe_j = V^{-1}Av_j =$$

$$= V^{-1}\lambda_j v_j = \lambda_j V^{-1} v_j = \lambda_j e_j$$

$$V^{-1}AV = \begin{array}{c} \\ \\ \\ \end{array}\begin{bmatrix} \lambda_1 & & & 0 & \\ & \lambda_2 & & \vdots & \\ & & \ddots & 0 & \\ & & & \lambda_j & \\ & & & 0 & \ddots \\ & & & \vdots & \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow \text{j-th row} \\ \\ \\ \end{array} = \Lambda.$$

j-th

$$\boxed{V^{-1}V = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}$$

j-th.

Def: If we can do the above
6.2.2. A is diagonalizable.

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Def 6.2.3. We say $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are similar matrices if there exists S $n\times n$ invertible s.t.

$$B = S^{-1} A S.$$

Prop 6.2.4. Similar matrices have the same eigenvalues.

Proof: HW (see Challenge 64 and 65)

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Theorem 6.3.1 [Spectral Theorem]

Any symmetric matrix $A \in \mathbb{R}^{n\times n}$, $A^T = A$ has a complete set of real eigenvectors and they can be made orthonormal.
(and has n real eigenvalues, up to repetition).

(if $B \in \mathbb{R}^{m\times n}$ not symmetric, $B^T B$ and $B B^T$ are symmetric)

Corollary 6.3.3.
If $A \in \mathbb{R}^{n \times n}$ is symmetric.

$$A = V \Lambda V^T \qquad (2)$$

where $V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ matrix with n orthonormal eigenvectors of A, and Λ diag with $\Lambda_{ii} = \lambda_i$.

Remarks 6.3.5. (1) and (2) and (3).
eigendecompositions.

Cor. 6.3.4. $A \in \mathbb{R}^{n \times n}$ sym,
rank(A) = # non-zero e-values of A
(counting repetitions)

Prop 6.3.6. $A \in \mathbb{R}^{n \times n}$ sym. $v_1, \dots, v_n$ orthonormal basis of e-vectors, $\lambda_1, \dots, \lambda_n$ corresponding e-values

$$A = \sum_{k=1}^{n} \underbrace{\lambda_k v_k v_k^T}_{\text{rank } 1} . \qquad (3)$$

HW: prove this.

Prop 6.3.7. Let $A \in \mathbb{R}^{n\times n}$ sym. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Proof: Let $v \in \mathbb{C}^n$ be an e-vector of A associated with λ.

$$Av = \lambda v$$

recall $M^* = \overline{M}^T$. since $A \in \mathbb{R}^{n\times n}$ and sym

$$A^* = \overline{A}^T = A^T = A$$

$$= v^* A v = v^*(\lambda v) = \lambda v^* v = \\ = \lambda \|v\|^2$$

$$= (\lambda v)^* v = (Av)^* v = v^* A^* v = v^* A v =$$

$$= \overline{\lambda}\, v^* v = \overline{\lambda} \|v\|^2$$

$$\lambda \|v\|^2 = \overline{\lambda} \|v\|^2$$

$$\Leftrightarrow \quad \lambda = \overline{\lambda}$$

$$\Leftrightarrow \quad \lambda \in \mathbb{R}.$$

Obs: True for $\underbrace{A \in \mathbb{C}^{n\times n} \text{ s.t. } A^* = A}_{A \text{ Hermitian.}}$

Cor 6.3.8 Every real symmetric matrix has a real eigenvalue.

Remark 6.3.9.

$A \in \mathbb{R}^{n\times n}$ sym.

v_1 and v_2 e-vectors of A with dif. e-values $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$\lambda_1 v_1^T v_2 = (\lambda_1 v_1)^T v_2 = (A v_1)^T v_2 = v_1^T A^T v_2 = v_1^T A v_2 = v_1^T \lambda_2 v_2$$
$$= \lambda_2 v_1^T v_2$$

$$\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$$

$$\lambda_1 \neq \lambda_2$$

$$\Longrightarrow v_1^T v_2 = 0.$$

$$v_1 \perp v_2.$$

Proof of Spectral Theorem.

$A \in \mathbb{R}^{n \times n}$ sym.

- For any $1 \leq k \leq n$, A has k orthonormal eigenvectors.

$k = n$ gives us our Thm.

Let's use induction.

→ $k = 1$. By Cor 6.3.8 there is a real eigenvector and we can normalize it to have norm 1.

→ Assuming we have $v_1, \dots, v_k$ orthonormal eigenvectors of A (with evalues $\lambda_1, \dots, \lambda_k$) we will show we can "add" one more v_{k+1}.

Let $u_{k+1}, \dots, u_n$ be an orthonormal basis for the orthogonal complement of the span of $v_1, \dots, v_k$.

$v_1, \dots, v_k, u_{k+1}, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$

V $n \times n$ matrix $V = \begin{bmatrix} | & & | & | & & | \\ v_1 & \cdots & v_k & u_{k+1} & \cdots & u_n \\ | & & | & | & & | \end{bmatrix}$

$V^T V = I$, $V V^T = I$.

$$B = V^T A V = \begin{bmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_k^T & — \\ — & u_{k+1}^T & — \\ & \vdots & \\ — & u_n^T & — \end{bmatrix} \begin{bmatrix} | & | & & | & | & \\ Av_1 & Av_2 & \cdots & Av_k & Au_{k+1} & \cdots \\ | & | & & | & | & \end{bmatrix}$$

$$= \begin{bmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_k^T & — \\ — & u_{k+1}^T & — \\ & \vdots & \\ — & u_n^T & — \end{bmatrix} \begin{bmatrix} | & & | & | & \\ \lambda_1 v_1 & \cdots & \lambda_k v_k & Au_{k+1} & \cdots \\ | & & | & | & \end{bmatrix} = \begin{bmatrix} \text{Nice} & \text{Nice} \\ \text{Nice} & \text{Terrible} \end{bmatrix}$$