D-INFK
Bernd Gärtner
Robert Weismantel

Linear Algebra

HS 2024

Assignment 10

Submission Deadline: **03 December, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Properties of pseudoinverses (in-class) (★★☆)

This exercise includes Challenge 15 from the lecture notes.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ be arbitrary matrices.

a) Prove that if $\text{rank}(A) = \text{rank}(B) = n$, we have $(AB)^\dagger = B^\dagger A^\dagger$.

b) Prove that $A^\dagger A A^\dagger = A^\dagger$.

c) Prove that $(A^\top)^\dagger = (A^\dagger)^\top$.

d) Prove that $A^\dagger A$ is symmetric and that it is the projection matrix for the subspace $\mathbf{C}(A^\top)$.

Hint: Use Proposition 5.5.9 from the lecture notes.

2. Challenge 16 (hand-in) (★★☆)

This task is Challenge 16 from the lecture notes which asks you to prove Proposition 5.5.12.

Let $A \in \mathbb{R}^{m \times n}$ be an arbitrary matrix with column space $\mathbf{C}(A)$ and row space $\mathbf{C}(A^\top)$. Consider the function $f : \mathbf{C}(A^\top) \to \mathbf{C}(A)$ that maps $\mathbf{x} \in \mathbf{C}(A^\top)$ to $(A\mathbf{x}) \in \mathbf{C}(A)$. Prove that f is bijective.

Hint: Solve Exercise 1 first.

3. Challenge 18 (★★☆)

This task is Challenge 18 from the lecture notes.

Let $A \in \mathbb{Q}^{m \times n}$, $\mathbf{b} \in \mathbb{Q}^m$, and $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \le \mathbf{b}\}$. Let $S_{n-k} = \{1, \ldots, n-k\}$ and $S_{n-j} = \{1, \ldots, n-j\}$ for indices $1 \le k < j < n$. Prove that

$$\mathsf{proj}_{S_{n-j}}(P) = \mathsf{proj}_{S_{n-j}}(\mathsf{proj}_{S_{n-k}}(P)).$$

4. Challenge 19 (★★☆)

This task is Challenge 19 from the lecture notes which asks you to prove parts of Theorem 5.6.6.

Recall the definition of $P^{(i)}$ from Definition 5.6.5. Prove that $P^{(i)} \subseteq \mathsf{proj}_{S_{n-i}}(P)$, where $S_{n-i} := \{1, \ldots, n-i\}$ for all $i \in [n]$.

5. Separating polyhedra in the plane with Farkas lemma (★★★)

Consider two polyhedra $P_1 \subseteq \mathbb{R}^2$ and $P_2 \subseteq \mathbb{R}^2$ in the plane. Assume that they do not intersect, i.e. $P_1 \cap P_2 = \emptyset$. Prove that there exists a vector $\mathbf{v} \in \mathbb{R}^2$ and a scalar w such that $P_1 \subseteq \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{v} \leq w\}$ and $P_2 \subseteq \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{v} > w\}$. Observe that intuitively speaking, this means that it is possible to draw a line that separates the two polyhedra.

Hint: *Exploit $P_1 \cap P_2 = \emptyset$ with Farkas lemma. Also note that there is nothing special about the plane, and you could solve this in higher dimensions as well (but assuming the setting of the plane might help with drawing and intuition).*