D-INFK
Bernd Gärtner
Robert Weismantel

Linear Algebra

HS 2024

Assignment 4

Submission Deadline: **22 October, 2024** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. **Solving linear systems (in-class)** (★☆☆) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most 2, i.e. $p(x) = ax^2 + bx + c$ for some coefficients $a, b, c \in \mathbb{R}$. Assume that we already know $p(-1) = 0$, $p(0) = 2$ and $p(1) = 2$. Find the coefficients a, b and c. As the title suggests, you will have to solve a linear system. We recommend that you do it by using the systematic elimination procedure from the lecture.

2. **Invertibility (hand-in)** (★☆☆)

   Let $A \in \mathbb{R}^{3\times 3}$ be the following upper triangular matrix with $a, b, c, d \in \mathbb{R}$:

   $$A = \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}.$$

   For which values of a, b, c, d is A invertible? Specify A^{-1} for these cases.

3. **Matrix inverse** (★☆☆)

   a) Consider the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and the standard unit vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find the solutions $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^3$ for the three systems $A\mathbf{x}_1 = \mathbf{e}_1$, $A\mathbf{x}_2 = \mathbf{e}_2$, $A\mathbf{x}_3 = \mathbf{e}_3$.

   b) What is the inverse A^{-1} of A?

   c) What is the inverse D^{-1} of the diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$?

   d) What is the inverse B^{-1} of the matrix $B = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$?

4. **Matrix inverse** (★★☆)

a) Let A be an $m \times m$ matrix with inverse A^{-1} and let $k \in \mathbb{N}^+$ be an arbitrary integer. Does A^k have an inverse and if yes, what is it?

b) Recall the definition of a nilpotent matrix: We say that a square matrix A is nilpotent if and only if there exists $k \in \mathbb{N}$ such that $A^k = 0$. Prove that a nilpotent matrix A cannot have an inverse.

c) Let A be an $m \times m$ matrix with $A^3 = I$ and $A^4 = I$. Prove that $A = I$.

d) Find a 2×2 matrix $A \neq I$ such that $A^k = I$ for all even k and $A^k = A$ for all odd $k \in \mathbb{N}$.

e) Can you also find a 2×2 matrix A that, for all $k \in \mathbb{N}$, satisfies $A^k = I$ if and only if $k \equiv_4 0$ (i.e. $A^k = I$ if and only if k is a multiple of 4)?

5. Exercise 3.12 (★★★)

Given two $m \times m$ matrices A and B with $AB = I$, prove that $BA = I$. You can either try to solve this directly or proceed according to the following subtasks, which should guide you through the proof.

a) Prove that the columns of B are linearly independent, i.e. B has rank m.

b) Use a) to prove that the columns of A are also linearly independent.

c) Use b) to prove that $BA - I = 0$.

6. Inverse of triangular matrices (★★★)

a) Find the inverse of the 2×2 matrix $L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ where $a \in \mathbb{R}$.

b) Prove that a square lower triangular matrix is invertible if and only if all its diagonal entries are non-zero.

c) Prove that the inverse of any lower triangular matrix, if it exists, is lower triangular itself.

d) Are the statements of b) and c) also true if we replace *lower triangular* by *upper triangular*?