D-INFK
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Linear Algebra

HS 2024

Assignment 5

Submission Deadline: **29 October, 2024** at 23:59

Course Website: https://ti.inf.ethz.ch/ew/courses/LA24/index.html

Exercises

You can get feedback from your TA for Exercise 3 by handing in your solution as pdf via Moodle before the deadline.

1. Elimination, back substitution, LU factorization (in-class) (★☆☆)

a) Compute an LU factorization of the matrix $A = \begin{bmatrix} 2 & -12 & 6 \\ 1 & -4 & 1 \\ 2 & -11 & -5 \end{bmatrix}$.

b) Given the factorization $A = LU$ from above, solve the linear system $L\mathbf{y} = \mathbf{b}$ with $\mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 25 \end{bmatrix}$.

c) Given the solution $\mathbf{y}$ to the linear system $L\mathbf{y} = \mathbf{b}$ above, solve the linear system $U\mathbf{x} = \mathbf{y}$ (for $\mathbf{x}$).

d) Given the solution $\mathbf{x}$ for the system above, prove that $A\mathbf{x} = \mathbf{b}$, i.e. $\mathbf{x}$ also solves this system.

2. Gauss-Jordan and $A = CR$ (in-class) (★☆☆)

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix}$$

with CR-decomposition

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} = CR.$$

Let R_0 be the matrix in row echolon form obtained by performing Gauss-Jordan elimination on A. Determine R_0 by performing the elimination and verify Theorem 3.24 by checking that R corresponds exactly to the non-zero rows of R_0 (i.e. R is the reduced row echolon form of A obtained by removing zero rows from R_0).

3. Inverse (hand-in) (★★☆)

a) Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

Justify your answer.

b) Find the inverse of the matrix $A \in \mathbb{R}^{m\times m}$ defined by

$$a_{ij} := \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in [m]$. Justify your answer. Note that this is a generalization of subtask a), where you were asked to solve the special case $m = 4$.

4. Vectors in $\mathbb{R}^4$ (★☆☆)

Consider the four vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -4 \\ 8 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 5 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ -5 \\ 5 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 6 \\ 2 \end{bmatrix}$$

in $\mathbb{R}^4$.

a) Is there a way to write $\mathbf{b}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

b) Are the three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ linearly independent?

***Hint**: You can solve both questions at once by computing a CR-decomposition.*

5. Interpolation (★☆☆) Assume that you gathered the following datapoints

x	y
0	1
2	2
4	5
6	6

and you want to find a function $f : \mathbb{R} \to \mathbb{R}$ that interpolates them, i.e. f should satisfy $f(x) = y$ for all pairs of x, y given by the table above. There is an abundance of functions that you can try and in particular, there are many different functions that do interpolate the four datapoints. In this exercise, we are interested in polynomials, i.e. we restrict f to be a polynomial of degree at most 3. In particular, this means that f has the form $f(x) = ax^3 + bx^2 + cx + d$ for some $a, b, c, d \in \mathbb{R}$. Your task is to find values for a, b, c, d such that f interpolates all four points given in the table.

6. Linear independence in $\mathbb{R}^3$ (★★☆)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be three linearly independent vectors in $\mathbb{R}^3$. Consider the three vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^3$ defined as

$$\mathbf{w}_1 := \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_2 := -\mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 := \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.$$

Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent.

7. Yet another decomposition (★★★)

Let $A \in \mathbb{R}^{m \times m}$ be a symmetric matrix with an LU-decomposition $A = LU$. In this exercise, we want to prove that there exists a diagonal matrix $D \in \mathbb{R}^{m \times m}$ such that $A = LDL^\top$.

a) Why is $L^\top$ invertible? (Recall Exercise 6 on Assignment 4.)

b) Define $D := U(L^\top)^{-1}$. Prove that D is upper triangular.

c) Prove that D satisfies $A = LDL^\top$.

d) Prove that D is symmetric.

e) Conclude that D is diagonal.