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Solution for Assignment 13

1. The matrix $-S^2$ is symmetric since

$$(-S^2)^\top = -(S^2)^\top = -(S^\top)^2 = -(-S)^2 = -S^2$$

where we used the assumption $S^\top = -S$.

From the lecture, we know that a symmetric matrix such as $-S^2$ is positive semidefinite if $\mathbf{x}^\top(-S^2)\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. To verify that this holds here, let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary and observe that

$$\mathbf{x}^\top(-S^2)\mathbf{x} = \mathbf{x}^\top(-S)S\mathbf{x} = \mathbf{x}^\top S^\top S\mathbf{x} = \|\, S\mathbf{x}\,\|^2 \geq 0.$$

We conclude that $-S^2$ is positive semidefinite.

2. Let $\mathbf{v} \in \mathbb{R}^n$ be an arbitrary non-zero vector. We calculate

$$\mathbf{v}^\top A\mathbf{v} = \sum_{i=1}^n \sum_{j=1}^n v_i v_j A_{ij} = n\sum_{i=1}^n v_i^2 + \sum_{i<j} 2v_iv_j \geq n\sum_{i=1}^n v_i^2 + \sum_{i<j}(-v_i^2 - v_j^2) = \sum_{i=1}^n v_i^2 > 0,$$

where we have used that $0 \leq (v_i + v_j)^2 = v_i^2 + 2v_iv_j + v_j^2$ for all $i, j \in [n]$. We conclude that A is indeed positive definite.

3. Consider first the $r \times n$ matrix $B = \Sigma_r V_r^\top$ with rank r. In particular, B has full row rank and hence

$$B^\dagger = B^\top(BB^\top)^{-1} = V_r\Sigma_r(\Sigma_r V_r^\top V_r \Sigma_r)^{-1} = V_r\Sigma_r(\Sigma_r^2)^{-1} = V_r\Sigma_r^{-1}$$

where we have used Definition 5.5.3, the fact that Σ_r is a diagonal matrix, and the fact that $V_r^\top V_r = I$.

Similarly, the $m \times r$ matrix U_r has full column rank r and hence we get

$$U_r^\dagger = (U_r^\top U_r)^{-1}U_r^\top = IU_r^\top = U_r^\top$$

by Definition 5.5.1 and the fact that $U_r^\top U_r = I$.

Finally, we conclude that

$$A^\dagger = B^\dagger U_r^\dagger = V_r\Sigma_r^{-1}U_r^\top$$

by Proposition 5.5.9.

4. **a)** The main idea is to plug in the SVD of A. A crucial observation that we will need is that by orthogonality of U, we have $\|\, U^\top\mathbf{v}\,\|_2^2 = (U^\top\mathbf{v})^\top(U^\top\mathbf{v}) = \mathbf{v}^\top UU^\top\mathbf{v} = \mathbf{v}^\top\mathbf{v} = \|\,\mathbf{v}\,\|_2^2$ for all $\mathbf{v} \in \mathbb{R}^m$. Equipped with this observation, we calculate

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2 &= \min_{\mathbf{x}\in\mathbb{R}^n} \|U\Sigma V^\top\mathbf{x} - \mathbf{b}\|_2^2 \\
&= \min_{\mathbf{x}\in\mathbb{R}^n} \|U^\top U\Sigma V^\top\mathbf{x} - U^\top\mathbf{b}\|_2^2 \\
&= \min_{\mathbf{x}\in\mathbb{R}^n} \|\Sigma V^\top\mathbf{x} - U^\top\mathbf{b}\|_2^2 \\
&= \min_{\mathbf{y}\in\mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2
\end{aligned}$$

where we have substituted $\mathbf{y} = V^\top\mathbf{x}$ in the end (which works because $V^\top$ is invertible).

b) Consider the expression $\|\Sigma\mathbf{y}-\mathbf{c}\|_2^2$ and observe that we can write it as

$$\|\Sigma\mathbf{y}-\mathbf{c}\|_2^2=\sum_{i=1}^{n}(\Sigma_{ii}y_i-c_i)^2=\sum_{i=1}^{r}(\sigma_i y_i-c_i)^2+\sum_{i=r+1}^{n}c_i^2.$$

We are looking to choose $\mathbf{y}$ such that this expression is minimized. Clearly, there is nothing that we can do about the term $\sum_{i=r+1}^{n}c_i^2$. But by choosing $y_i=c_i/\sigma_i$ for all $i\in[r]$, we get $\sum_{i=1}^{r}(\sigma_i y_i-c_i)^2=0$. Hence, this choice of $\mathbf{y}$ must be optimal. Concretely, we conclude that the optimal solution is

$$\mathbf{y}^*=\begin{bmatrix} c_1/\sigma_1\\ \vdots\\ c_r/\sigma_r\\ 0\\ \vdots\\ 0\end{bmatrix}=\arg\min_{\mathbf{y}\in\mathbb{R}^n}\|\Sigma\mathbf{y}-\mathbf{c}\|_2^2.$$

c) In subtask a), we substituted $\mathbf{y}=V^\top\mathbf{x}$. Hence, it would make sense to guess that $\mathbf{x}^*=V\mathbf{y}^*$. Indeed, we can verify that with this choice of $\mathbf{x}^*$ we get

$$\|\Sigma\mathbf{y}^*-\mathbf{c}\|_2^2=\|\Sigma V^\top\mathbf{x}^*-\mathbf{c}\|_2^2=\|U\Sigma V^\top\mathbf{x}^*-UU^\top\mathbf{b}\|_2^2=\|U\Sigma V^\top\mathbf{x}^*-UU^\top\mathbf{b}\|_2^2=\|A\mathbf{x}^*-\mathbf{b}\|_2^2$$

and by $\min\limits_{\mathbf{x}\in\mathbb{R}^n}\|A\mathbf{x}-\mathbf{b}\|_2^2=\min\limits_{\mathbf{y}\in\mathbb{R}^n}\|\Sigma\mathbf{y}-\mathbf{c}\|_2^2$ and optimality of $\mathbf{y}^*$ we conclude that $\mathbf{x}^*$ is optimal, i.e.

$$\mathbf{x}^*=\arg\min_{\mathbf{x}\in\mathbb{R}^n}\|A\mathbf{x}^*-\mathbf{b}\|_2^2.$$

5. **a)** We prove this by direct calculation

$$\|\,\mathbf{x}\,\|_2^2=\sum_{i=1}^{n}x_i^2\le\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i||x_j|=(\sum_{i=1}^{n}|x_i|)^2=\|\,\mathbf{x}\,\|_1^2\,.$$

Observe that the inequality $\sum_{i=1}^{n}x_i^2\le\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i||x_j|$ holds because all terms appearing on the left actually appear on the right as well (but on the right we have some additional non-negative terms).

b) Without loss of generality, assume that all entries in $\mathbf{x}$ are non-negative (if there was a negative entry, simply switch its sign and observe that both norms remain the same). Next, observe that $\|\,\mathbf{x}\,\|_1=\sum_{i=1}^{n}|x_i|=\sum_{i=1}^{n}x_i=\mathbf{1}^\top\mathbf{x}$ where $\mathbf{1}\in\mathbb{R}^n$ is the all-ones vector. By Cauchy-Schwarz, we obtain $\mathbf{1}^\top\mathbf{x}\le\|\,\mathbf{1}\,\|_2\,\|\,\mathbf{x}\,\|_2$. It remains to calculate $\|\,\mathbf{1}\,\|_2=(\sum_{i=1}^{n}1)^{\frac{1}{2}}=\sqrt{n}$ to conclude that

$$\|\,\mathbf{x}\,\|_1=\mathbf{1}^\top\mathbf{x}\le\|\,\mathbf{1}\,\|_2\,\|\,\mathbf{x}\,\|_2=\sqrt{n}\,\|\,\mathbf{x}\,\|_2\,.$$

6. **a)** Recall that the trace of a matrix is the sum of its diagonal entries. Consider the matrix $A^\top A$. The j-th diagonal entry of $A^\top A$ is exactly the norm of the j-th column of A which is given by $\sum_{i=1}^{m}A_{ij}^2$. Hence, the trace of $A^\top A$ is given by

$$\mathrm{Tr}(A^\top A)=\sum_{j=1}^{n}\sum_{i=1}^{m}A_{ij}^2=\sum_{i=1}^{m}\sum_{j=1}^{n}A_{ij}^2=\|\,A\,\|_F^2\,.$$

b) We know that the squared singular values of A are the eigenvalues of the matrix $A^\top A$. Moreover, we know that the trace of $A^\top A$ is equal to the sum of its eigenvalues. Hence, we conclude

$$\text{Tr}(A^\top A) = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2$$

and the result follows by combining this with the previous subtask.

c) By definition, we have

$$\| A \|_{op} = \max_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \| \mathbf{x} \|_2 = 1}} \| A\mathbf{x} \|_2 .$$

Now observe that we can rewrite the squared version of this as

$$\max_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \| \mathbf{x} \|_2^2 = 1}} \| A\mathbf{x} \|_2^2 = \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{0\}} \frac{\| A\mathbf{x} \|_2^2}{\| \mathbf{x} \|_2^2} = \max_{\mathbf{x}\in\mathbb{R}^n\setminus\{0\}} \frac{\mathbf{x}^\top A^\top A\mathbf{x}}{\mathbf{x}^\top\mathbf{x}}.$$

The matrix $A^\top A$ is symmetric and its largest eigenvalue is σ_1^2, hence we get $\max_{\mathbf{x}\in\mathbb{R}^n\setminus\{0\}} \frac{\mathbf{x}^\top A^\top A\mathbf{x}}{\mathbf{x}^\top\mathbf{x}} = \sigma_1^2$ by Proposition 7.3.10. It remains to observe that

$$\arg\max_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \| \mathbf{x} \|_2 = 1}} \| A\mathbf{x} \|_2 = \arg\max_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \| \mathbf{x} \|_2^2 = 1}} \| A\mathbf{x} \|_2^2$$

and hence

$$\| A \|_{op} = \max_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \| \mathbf{x} \|_2 = 1}} \| A\mathbf{x} \|_2 = \sqrt{\max_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \| \mathbf{x} \|_2^2 = 1}} \| A\mathbf{x} \|_2^2} = \sqrt{\sigma_1^2} = \sigma_1.$$

d) This follows from b) and c) as

$$\| A \|_{op} = \sigma_1 = \sqrt{\sigma_1^2} \leq \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2} = \| A \|_F .$$

e) Using previous subtasks, we obtain

$$\| A \|_F^2 = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2 \leq \min\{m,n\}\sigma_1^2$$

and hence

$$\| A \|_F \leq \sqrt{\min\{m,n\}}\sigma_1 = \sqrt{\min\{m,n\}} \| A \|_{op} .$$