Solution for Assignment 5

1. **a)** We will use the elimination procedure on A in order to get the upper triangular matrix U and the lower triangular matrix L. First, we multiply A with

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

to get

$$E_{31}E_{21}A = \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 1 & -11 \end{bmatrix}.$$

We also note down the coefficients $\ell_{21} = \frac{1}{2}$ and $\ell_{31} = 1$ for L. Note that these are just the negated entries of E_{21} and E_{31}, respectively. Next, we multiply this with $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$ and get

$$E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & -10 \end{bmatrix} =: U$$

which is upper triangular. We also write down $\ell_{32} = \frac{1}{2}$. From the lecture we know that we can obtain L as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix}.$$

Indeed, checking

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 2 & -12 & 6 \\ 1 & -4 & 1 \\ 2 & -11 & -5 \end{bmatrix} = A$$

we conclude that this is a valid LU factorization of A.

b) Since L is lower triangular, we can start substituting from the top. In particular, writing out the equation gives

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 4 \\ 4 \\ 25 \end{bmatrix} = \mathbf{b}.$$

Hence, we get $y_1 = 4$, $y_2 = 4 - 2 = 2$, and $y_3 = 25 - 1 - 4 = 20$.

c) We first write down the system again as

$$U\mathbf{x} = \begin{bmatrix} 2 & -12 & 6 \\ 0 & 2 & -2 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \overset{!}{=} \begin{bmatrix} 4 \\ 2 \\ 20 \end{bmatrix} = \mathbf{y}.$$

By using back substitution we obtain $x_3 = \frac{20}{-10} = -2$, $x_2 = \frac{2-4}{2} = -1$, and $x_1 = \frac{4-12+12}{2} = 2$.

d) Using the results from the previous two subtasks we get

$$A\mathbf{x} \overset{LU}{=} LU\mathbf{x} = L(U\mathbf{x}) \overset{\text{c)}}{=} L\mathbf{y} \overset{\text{b)}}{=} \mathbf{b}.$$

2. We perform Gauss-Jordan elimination on

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 2 & 5 \end{bmatrix}.$$

Our first pivot is already 1, so we can eliminate in the first column to get

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 2 & -4 \end{bmatrix}.$$

The second column is already done so we move on to eliminate in the third column. Luckily, our pivot is already 1 again so we do not have to normalize. We obtain

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and observe that we are done since this matrix is in row echolon form. In other words, this is our R_0 and we can check that it corresponds exactly to R on the first two rows.

3. **a)** The inverse of A is $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. To justify this, it suffices to check that AA^{-1} indeed equals I. But let us still explain how we found A^{-1}: Finding A^{-1} can be done by finding vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ (columns of A^{-1}) such that

$$A\mathbf{v}_i = \mathbf{e}_i$$

for all $i \in \{1, 2, 3, 4\}$, where $\mathbf{e}_i$ is the i-th standard unit vector. Using e.g. elimination to solve these systems, we get

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and thus $A^{-1} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Alternatively, one might also be able to guess the vectors $\mathbf{v}_1, \dots, \mathbf{v}_4$ by noticing that by subtracting the $(i+1)$-th column of A from the i-th column of A, we get $\mathbf{e}_i$, for all $i \in \{1, 2, 3\}$ (and that $\mathbf{v}_4 = \mathbf{e}_4$).

b) We solve this exercise by guessing that the pattern from a) also works in general. Concretely, we define the matrix $B \in \mathbb{R}^{m \times m}$ with columns $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^m$ such that $\mathbf{v}_i := \mathbf{e}_i - \mathbf{e}_{i+1}$ for all $i \in \{1, 2, \dots, m-1\}$ and $\mathbf{v}_m := \mathbf{e}_m$. We claim that B is the unique inverse of A. To prove this, first observe that the i-th row of A is given by $\sum_{k=1}^{i} \mathbf{e}_k^\top$. This means that the entry $(AB)_{ij}$ is given by

$$(AB)_{ij} = \big(\sum_{k=1}^{i} \mathbf{e}_k^\top\big)\mathbf{v}_j = \sum_{k=1}^{i} \mathbf{e}_k^\top \mathbf{v}_j$$

for all $i, j \in [m]$. Let now $i, j \in [m]$ be arbitrary. We distinguish three cases.

- Assume first $j < i$. Then we get $\sum_{k=1}^{i} \mathbf{e}_k^\top \mathbf{v}_j = \sum_{k=1}^{i} \mathbf{e}_k^\top (\mathbf{e}_j - \mathbf{e}_{j+1}) = \mathbf{e}_j^\top \mathbf{e}_j - \mathbf{e}_{j+1}^\top \mathbf{e}_{j+1} = 0$.
- Next, assume $j > i$. Observe that in this case, we have $\mathbf{e}_k^\top \mathbf{v}_j = 0$ for all $k \in [i]$ and thus $(AB)_{ij} = 0$.
- Finally, we observe that $\sum_{k=1}^{i} \mathbf{e}_k^\top \mathbf{v}_j = \mathbf{e}_j^\top \mathbf{e}_j = 1$ for $i = j$.

We conclude that $AB = I$ and thus B is the unique inverse of A.

4. We solve both subtasks at once using the CR-decomposition. Consider the matrix

$$\begin{bmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{b} \\ | & | & | & | \end{bmatrix}$$

obtained by using $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{b}$ as columns. Concretely, we have

$$\begin{bmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{b} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 & 1 \\ -4 & 5 & -5 & -2 \\ 8 & 5 & 5 & 6 \\ 2 & 2 & 1 & 2 \end{bmatrix}.$$

We compute the CR-decomposition of this matrix. After dividing the first row by 2 and eliminating the first column, we get

$$\begin{bmatrix} 1 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 3 & -1 & 0 \\ 0 & 9 & -3 & 2 \\ 0 & 3 & -1 & 1 \end{bmatrix}.$$

Next, we divide the second row by 3 and eliminate in the second column to get

$$\begin{bmatrix} 1 & 0 & \frac{5}{6} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally, we divide the third row by 2 and eliminate in the fourth column to obtain

$$\begin{bmatrix} 1 & 0 & \frac{5}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We conclude that our matrix has CR-decomposition

$$\begin{bmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{b} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{b} \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{5}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Interpreting this result, we deduce that $\mathbf{b}$ cannot be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ since we have a pivot in the fourth column. Moreover, the reduced row echolon form also shows that $\mathbf{u}_3$ is dependent on $\mathbf{u}_1$ and $\mathbf{u}_2$. Hence, the three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent.

5. Each of the four points yields one linear equation with variables a, b, c, d. For example, for $x = 4, y = 5$ we get the equation

$$a4^3 + b4^2 + c4 + d = 5.$$

In total, we get the linear system

$$\begin{aligned} a0^3 + b0^2 + c0 + d &= 1 \\ a2^3 + b2^2 + c2 + d &= 2 \\ a4^3 + b4^2 + c4 + d &= 5 \\ a6^3 + b6^2 + c6 + d &= 6 \end{aligned}$$

with four equations and four variables that we can write in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 6 & 36 & 216 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 6 \end{bmatrix}.$$

We now want to solve this system by using the elimination technique. For this, it is convenient to apply the row operations to the system matrix and the right-hand side simultaneously by appending the right-hand side to the matrix as follows:

$$\left[\begin{array}{cccr|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 4 & 8 & 2 \\ 1 & 4 & 16 & 64 & 5 \\ 1 & 6 & 36 & 216 & 6 \end{array}\right].$$

After performing elimination in the first column we get

$$\left[\begin{array}{cccr|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 4 & 16 & 64 & 4 \\ 0 & 6 & 36 & 216 & 5 \end{array}\right].$$

Next, we perform elimination in the second columns to get

$$\left[\begin{array}{cccr|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 24 & 192 & 2 \end{array}\right].$$

Finally, we obtain

$$\left[\begin{array}{cccr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 0 & 48 & -4 \end{array}\right].$$

It remains to perform backward substitution. From the last row, we get $a = -\frac{4}{48} = -\frac{1}{12}$. Next, we get $b = \frac{2-48a}{8} = \frac{6}{8} = \frac{3}{4}$. From the second row we obtain $c = \frac{1-8a-4b}{2} = \frac{1+\frac{2}{3}-3}{2} = -\frac{2}{3}$. Finally, we get $d = 1$ from the first row. Hence, the function $f(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 1$ interpolates all of our datapoints.

6. We want to prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent. Consider the matrices

$$W := \begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \\ | & | & | \end{bmatrix}, V := \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}, \text{ and } M := \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that we have chosen M such that by definition of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, we have $W = VM$. Observe first that V has rank 3 and is invertible, since its columns are linearly independent (Inverse Theorem).

Next, we compute the rank of M. From the lecture, we know that the rank of a matrix is equal to the number of pivots after using Gauss elimination on the matrix. We use this on M: subtracting the first row of M once from its second row, we get the triangular matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which means that M has rank 3 as well. In other words, the columns of M are linearly independent and hence M is invertible.

By Lemma 3.9, we conclude that W is invertible as it can be written as the product of two invertible matrices. By the Inverse Theorem, the columns of W are linearly independent, as desired.

7. We will use statements from Exercise 6 on Assignment 4 without reproving them.

a) We know that L is a square lower triangular matrix with 1's on the diagonal. By Exercise 6 on Assignment 4, it follows that L is invertible. From the lecture we know that this also implies invertibility of $L^\top$.

b) Observe that $L^\top$ is upper triangular. Using Exercise 6 on Assignment 4, we conclude that $(L^\top)^{-1}$ is also upper triangular. Moreover, U is upper triangular. It remains to observe that the product of two upper triangular matrices is upper triangular again: Indeed, let $i, j \in [m]$ be arbitrary with $i > j$ and consider the entry D_{ij}. Let $\mathbf{v} \in \mathbb{R}^m$ be the i-th row of U, and let $\mathbf{w} \in \mathbb{R}^m$ be the j-th column of $(L^\top)^{-1}$. By definition of matrix multiplication, we have $D_{ij} = \mathbf{v} \cdot \mathbf{w}$. But since the first $i - 1$ entries of $\mathbf{v}$ are zero, the last $m - j$ entries of $\mathbf{w}$ are zero, and $i > j$, we can conclude $D_{ij} = \mathbf{v} \cdot \mathbf{w} = 0$. Thus, D is upper triangular.

c) Plugging in the definition of D, we indeed get $LDL^\top = LU(L^\top)^{-1}L^\top = LU = A$.

d) Since L is invertible, we can rewrite $LDL^\top = A$ to $D = L^{-1}A(L^\top)^{-1}$. We further observe that

$$D^\top = \left(L^{-1}A(L^\top)^{-1}\right)^\top = L^{-1}A^\top(L^\top)^{-1} = L^{-1}A(L^\top)^{-1}$$

where we used symmetry of A in the last step. Plugging in $A = LU$ and using $L^{-1}L = I$, we get $D^\top = U(L^\top)^{-1} = D$. In other words, D is symmetric as well.

e) We know that D is upper triangular from subtask b). By symmetry, this implies that D is lower triangular as well. We conclude that D is diagonal.