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Solution

1 Calculations I

For each subtask we provide you with a possible solution as well as pointers to the relevant learning goals and related exercises. Moreover, we provide an explanation for each subtask. Note that in this exercise, the explanation is not needed to get the points (we are only looking for the correct answer).

a) Canonical (but not unique) solution:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 12 \end{bmatrix}.$$

Learning Goals*: derive the $PA = LU$ factorization (week 4).*

Related Exercise*: Assignment 4, Exercise 4.*

Explanation*: We proceed with Gauss elimination for the matrix A. In the first elimination step, we add the first row of A three times to the second row of A. We also subtract the first row $\frac{1}{2}$ times from the third row. This yields the matrix*

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 12 \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Next, we swap the second and third row to obtain

$$U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 12 \end{bmatrix}.$$

Retracing our steps, we observe that we have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

because we swapped the second and third row during the elimination procedure. Finally, we can either compute $L = PAU^{-1}$ or remember that we can deduce L from the coefficients we used as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Here it is important to remember that we swapped the second and third row and hence the $\frac{1}{2}$ appears in the second row of L while -3 appears in the third row of L.

b) Canonical (but not unique) solution:

$$\begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Learning Goals*: compute a basis for the nullspace of a matrix in rref (week 5).*

Related Exercise*: Assignment 6, Exercise 1.*

Explanation*: The matrix M is already in reduced row echolon form. We can see that the second and fourth variable are free. Hence, we want to find 2 linearly independent vectors $\mathbf{x}, \mathbf{y} \in \mathbf{N}(M)$. A canonical way to do this is to pick $x_2 = 0, x_4 = 1$ and deduce x_1 and x_3 from that. This yields $\mathbf{x} = \begin{bmatrix} -2 & 0 & 3 & 1 \end{bmatrix}^\top$. Analogously, we pick $y_2 = 1$ and $y_4 = 0$ and solve for y_1 and y_3 to get $\mathbf{y} = \begin{bmatrix} 5 & 1 & 0 & 0 \end{bmatrix}^\top$.*

c) Canonical (but not unique) solution:

$$\frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Learning Goals*: build an orthonormal basis with Gram-Schmidt (week 8).*

Related Exercise*: Assignment 8, Exercise 1.*

Explanation*: We observe that the two columns of N are linearly independent. To transform them into an orthonormal basis, we use the Gram-Schmidt procedure. In a first step, we normalize the first column to get*

$$\frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}.$$

In a next step, we compute a vector orthogonal to the first column by subtracting the projection of the second column onto the first column from the second column to get:

$$\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} - \left(\frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \right) \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Finally, we normalize this vector to obtain the second vector of our orthonormal basis

$$\frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

2 Calculations II

a) Unique solution: $a = 2, b = -\frac{5}{3}$.

***Learning Goals**: use Least Squares to fit a line to points (week 8).*

***Related Exercise**: Quiz "Fitting a line to a more complex model" (week 8). Assignment 7, Exercise 4. Assignment 8, Exercise 4.*

***Explanation**: We want to solve the linear system*

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \approx \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

in the least squares sense. Pluggin in $x_1 = -\frac{1}{2}, x_2 = 0, x_3 = \frac{1}{2}$ as well as $y_1 = -\frac{5}{2}, y_2 = -2, y_3 = -\frac{1}{2}$ we get

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} \approx \begin{bmatrix} -\frac{5}{2} \\ -2 \\ -\frac{1}{2} \end{bmatrix}.$$

The least square solution can be obtained by solving the normal equations

$$\begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{5}{2} \\ -2 \\ -\frac{1}{2} \end{bmatrix}.$$

In particular, simplifying this we get the system

$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

and hence $a = 2$ and $b = -\frac{5}{3}$.

b) One possible solution:

$$\lambda_1 = 5, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 0, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}.$$

***Learning Goals**: eigenvalues and eigenvectors, characteristic polynomial, finding eigenvalues and eigenvectors, linear independence of eigenvectors corresponding to distinct eigenvalues (week 11).*

***Related Exercise**: Quizzes "Eigenvalue and eigenvectors (1)" and "Eigenvalue and eigenvectors (2)" (week 11).*

***Explanation**: We first find the eigenvalues by determining the roots of the characteristic polynomial. The characteristic polynomial is given by*

$$p_M(\lambda) = \mathsf{det}(M - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 3 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + 8\lambda^2 - 15\lambda = -\lambda(\lambda - 3)(\lambda - 5)$$

and hence has roots $\lambda_1 = 5, \lambda_2 = 3, \lambda_3 = 0$. It remains to find an eigenvector for each of the eigenvalues. Since all eigenvalues are distinct, the resulting eigenvectors are guaranteed to be linearly independent.

We can find $\mathbf{v}_3$ in $\mathbf{N}(M - 0I) = \mathbf{N}(M)$ e.g. with Gauss elimination. Concretely, the elimination yields

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence a possible solution is $\mathbf{v}_3 = \begin{bmatrix} -2 & 0 & 3 \end{bmatrix}^\top$.

Analogously, we can find $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbf{N}(M - 5I)$ and $\mathbf{N}(M - 3I)$, respectively.

3 Proofs I

a) We already know that $S \subseteq \mathbb{R}^n$. From the lecture, we know that a subset of a vector space is a subspace if it is non-empty and closed under vector addition and scalar multiplication. Let us prove that S satisfies these conditions.

The first and last coordinate of the all-zero vector $\mathbf{0} \in \mathbb{R}^n$ are both zero and hence we have $\mathbf{0} \in S$. We conclude that S is not empty and it remains to prove that it is closed under vector addition and scalar multiplication. Let $\mathbf{x}, \mathbf{y} \in S$ and $c \in \mathbb{R}$ be arbitrary. We need to prove $\mathbf{x} + \mathbf{y} \in S$ and $c\mathbf{x} \in S$.

The n-th coordinate of $\mathbf{x} + \mathbf{y}$ is given by $x_n + y_n$. Similarly, the first coordinate of $\mathbf{x} + \mathbf{y}$ is given by $x_1 + y_1$. Moreover, by $\mathbf{x}, \mathbf{y} \in S$ we have $x_1 = x_n$ and $y_1 = y_n$. We conclude that $x_n + y_n = x_1 + y_1$ and hence $(\mathbf{x} + \mathbf{y}) \in S$.

The n-th coordinate of the vector $c\mathbf{x}$ is given by cx_n. Similarly, its first coordinate is given by cx_1. By $x_1 = x_n$ we also get $cx_1 = cx_n$ and hence $(c\mathbf{x}) \in S$.

Since S is non-empty and closed under vector addition and scalar multiplication, we conclude that S is a subspace of $\mathbb{R}^n$.

***Learning Goals**: define and identify subspaces (week 4).*

***Related Exercise**: Assignment 4, Exercise 1. Assignment 5, Exercise 4.*

b) We know from the lecture that any three linearly independent vectors in $\mathbb{R}^3$ are a basis of $\mathbb{R}^3$. Hence, it remains to prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent. Consider the matrices

$$W := \begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \\ | & | & | \end{bmatrix}, V := \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}, \text{ and } M := \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that we have chosen M such that by definition of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, we have $W = VM$. Also observe that V has rank 3 since its columns span all of $\mathbb{R}^3$.

Next, we compute the rank of M. From the lecture, we know that the rank of a matrix is equal to the number of pivots after using Gauss elimination on the matrix. We use this on M: subtracting the first row of M once from its second row, we get the triangular matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which means that M has rank 3 as well.

We conclude that both V and M have rank 3 (and are invertible by the Inverse Theorem). From the lecture, we know that the product of two square invertible matrices is invertible again. This implies that W has rank 3 as well which again implies that the columns of W are linearly independent (Inverse Theorem) which is what we wanted to prove.

***Learning Goals**: explain when vectors span a subspace / form a basis of it (week 4). Define linear independence of vectors in three different ways (week 1). Change of basis, change of basis matrix (week 12).*

***Related Exercise**: Assignment 1, Exercise 1. Assignment 5, Exercise 2. Assignment 5, Exercise 5.*

c) From the lecture, we know that the rank of a matrix corresponds to the dimension of its column space. Moreover, the dimension of the column space and nullspace of a matrix add up to the number of columns. Concretely, we hence get

$$\text{rank}(A) = \dim(\mathbf{C}(A)) = 4 - \dim(\mathbf{N}(A))$$

and rank$(B) = \dim(\mathbf{C}(B))$. By the assumption $\mathbf{C}(B) \subseteq \mathbf{N}(A)$, we get $\dim(\mathbf{C}(B)) \leq \dim(\mathbf{N}(A))$. We conclude that

$$\text{rank}(A) + \text{rank}(B) = 4 - \dim(\mathbf{N}(A)) + \dim(\mathbf{C}(B)) \leq 4.$$

which is what we wanted to prove.

***Learning Goals**: define the four fundamental subspaces of a matrix: column space, row space, nullspace, left nullspace; compute their dimensions, depending on shape and rank of the matrix (week 6).*

***Related Exercise**: Assignment 6, Exercise 1. Assignment 6, Exercise 6. Assignment 5, Exercise 1.*

4 Proofs II

a) The matrix $-S^2$ is symmetric since

$$(-S^2)^\top = -(S^2)^\top = -(S^\top)^2 = -(-S)^2 = -S^2$$

where we used the assumption $S^\top = -S$.

From the lecture, we know that a symmetric matrix such as $-S^2$ is positive semidefinite if $\mathbf{x}^\top(-S^2)\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. To verify that this holds here, let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary and observe that

$$\mathbf{x}^\top(-S^2)\mathbf{x} = \mathbf{x}^\top(-S)S\mathbf{x} = \mathbf{x}^\top S^\top S\mathbf{x} = \| S\mathbf{x} \|^2 \geq 0.$$

We conclude that $-S^2$ is positive semidefinite.

***Learning Goals**: compute with matrices (week 1). Define and work with symmetric matrices (week 4). Positive semidefinite matrices (week 13).*

***Related Exercise**: Assignment 4, Exercise 5. Assignment 13, Exercise 1.*

b) No, f is not a linear transformation. Consider the standard unit vector $\mathbf{e}_n \in \mathbb{R}^n$ and choose $c = 2$. By definition of f, we get

$$f(c\mathbf{e}_n) = \sum_{k=1}^{n-1} c^k 0 + c^n = c^n$$

and also

$$f(\mathbf{e}_n) = \sum_{k=1}^{n-1} 0 + 1 = 1.$$

Hence, we have $f(c\mathbf{e}_n) = c^n = 2^n \neq 2 = cf(\mathbf{e}_1)$ which means that f is not a linear transformation. Note that the assumption $n \geq 2$ is crucial for this last step to work.

***Learning Goals**: linear transformations, examples of transformations that are linear, examples of transformations that are not linear (week 10).*

***Related Exercise**: Assignment 10, Exercise 3.*

c) Consider the matrix $A^\top A \in \mathbb{R}^{n\times n}$. From the lecture, we know that the matrix product $A^\top A$ is symmetric and that the real eigenvalues of $A^\top A$ (all of them are real) correspond to the squared singular values of A. Now observe that

$$A^\top A\mathbf{v} = A^\top \mathbf{w} = \mathbf{v}.$$

Since $\mathbf{v} \neq \mathbf{0}$, this implies that $A^\top A$ has eigenvalue 1 with corresponding eigenvector $\mathbf{v}$. This implies that A has singular value 1 (1 is the only non-negative real number with $1^2 = 1$, singular values are non-negative).

***Learning Goals**: singular value decomposition (SVD), derivation of singular value decomposition, connection to eigenvalue decomposition of $A^\top A$ and $AA^\top$ (week 13). Singular values, left singular vectors, right singular vectors (week 13).*

***Related Exercise**: Assignment 12, Exercise 3. Assignment 13, Exercise 2. Assignment 13, Exercise 3.*

***Explanation**: This exercise is relatively similar to Exercise 3 of Assignment 12, even though singular values do not make an explicit appearance there. You could also solve this exercise by using the SVD, which would be a bit closer to solutions of Exercises 2 and 3 of Assignment 13.*

Solution sheet for multiple choice questions

/24

Note: only the solutions you marked here will count.

Question No.	Answer			
	(a)	**(b)**	**(c)**	**(d)**
1	a	b	c	d
2	a	b	c	d
3	a	b	c	d
4	a	b	c	d
5	a	b	c	d
6	a	b	c	d

Circle the correct answer to each question in the table above. Only the answers in the table are considered, all other markings (for example in the text of the multiple choice questions themselves) will be ignored and not considered. Exactly one of the four answers to each question is correct. Each question earns 4 points if and only if it is solved correctly (meaning that the correct answer is circled). Wrong answers earn 0 points.

5 Multiple choice questions

Note that the difficulty of the multiple choice questions varies despite all of them being worth the same amount of points. Concretely, each of the following 6 multiple choice questions is worth 4 points.
Remember to give your answers on the **solution sheet** for multiple choice questions! We will only look at the solution sheet. In particular, anything you write directly into the multiple choice questions here will not be considered.

1. Assume that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ are linearly independent. Which of the following sets of vectors is linearly **dependent**? ***Note: only one answer is correct.***

(a) $\mathbf{v}_1,\ -2\mathbf{v}_2,\ \mathbf{v}_3$

(b) $\mathbf{v}_2,\ \mathbf{v}_4$

(c) $\mathbf{v}_1,\ 2\mathbf{v}_1 + \mathbf{v}_4,\ \mathbf{v}_3 + \mathbf{v}_4$

$\checkmark$ **(d)** $2\mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{v}_1 - 2\mathbf{v}_2,\ \mathbf{v}_4,\ 7\mathbf{v}_1 - 4\mathbf{v}_2$

Learning Goals: define linear independence of vectors in three different ways (week 1).
Explanation:

(a) Let $a, b, c \in \mathbb{R}$ be scalars such that $a\mathbf{v}_1 + b(-2\mathbf{v}_2) + c\mathbf{v}_3 = \mathbf{0}$. From this equality, we derive that $a\mathbf{v}_1 + -2b\mathbf{v}_2 + c\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$, and since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent, this implies that $a = 0$, $-2b = 0$, and $c = 0$. Therefore, we have proved that $a\mathbf{v}_1 + b(-2\mathbf{v}_2) + c\mathbf{v}_3 = 0$ implies $a = b = c = 0$, which means that the vectors $\mathbf{v}_1, -2\mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

(b) A subset of a set of linearly independent vectors is also linearly independent.

(c) Let $a, b, c \in \mathbb{R}$ be scalars such that $a\mathbf{v}_1 + b(2\mathbf{v}_1 + \mathbf{v}_4) + c(\mathbf{v}_3 + \mathbf{v}_4) = \mathbf{0}$, i.e. $(a + 2b)\mathbf{v}_1 + 0\mathbf{v}_2 + c\mathbf{v}_3 + (c + b)\mathbf{v}_4 = 0$. Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent, this implies

$$\begin{cases} a + 2b &= 0 \\ c &= 0 \\ c + b &= 0 \end{cases}$$

from which we deduce easily that $a = b = c = 0$, which means that the three vectors $\mathbf{v}_1, 2\mathbf{v}_1 + \mathbf{v}_4, \mathbf{v}_3 + \mathbf{v}_4$ are linearly independent.

(d) Notice that $7\mathbf{v}_1 - 4\mathbf{v}_2 = 2(2\mathbf{v}_1 + \mathbf{v}_2) + 3(\mathbf{v}_1 - 2\mathbf{v}_2)$, hence the set of vector is not linearly independent.

2. Let $a, b \in \mathbb{R}$ and consider the matrix $M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix}$. Assume that M is orthogonal. Which of the following statements must be **true**? ***Note: only one answer is correct.***

√ **(a)** $ab = -\frac{1}{2}$.

(b) $M^2 = I$.

(c) $a = \frac{b}{2}$.

(d) $b = \sqrt{1 - a^2}$.

Learning Goals: orthogonal matrices (week 8).
Explanation: M is orthogonal if and only if $M^\top M = I$. We compute that for all $a, b \in \mathbb{R}$

$$M^\top M = \begin{bmatrix} \frac{1}{\sqrt{2}} & a \\ \frac{1}{\sqrt{2}} & b \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + a^2 & \frac{1}{2} + ab \\ \frac{1}{2} + ab & \frac{1}{2} + b^2 \end{bmatrix}.$$

(a) Because we must have $M^\top M = I$, from the top right and bottom left elements we deduce $ab = -\frac{1}{2}$.

(b) If $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$, then $M^\top M = I$ but $M^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(c) If $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$, then $M^\top M = I$ but $a \neq -\frac{b}{2}$.

(d) If $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$, then $M^\top M = I$ but $b = -\sqrt{1 - a^2}$.

3. For two matrices $A, B \in \mathbb{R}^{n\times n}$ with $n \in \mathbb{N}^+$, which of the following statements must be **true**? ***Note: only one answer is correct.***

(a) If A and B have the same set of eigenvalues, then $A - B$ has an eigenvalue 0.

√ **(b)** AB and BA always have the same set of real eigenvalues.

(c) If $\mathbf{v} \in \mathbb{C}^n$ is an eigenvector of A corresponding to eigenvalue $\lambda \in \mathbb{C}$, then $\mathbf{v}$ is also an eigenvector of $A^\top$ corresponding to eigenvalue λ.

(d) If $\mathbf{v} \in \mathbb{R}^n$ is an eigenvector of A that corresponds to eigenvalue $\lambda \in \mathbb{R}$, and $\mathbf{v} \in \mathbb{R}^n$ is also an eigenvector of B that corresponds to eigenvalue 2λ, then $\mathbf{v} \in \mathbb{R}^n$ is also an eigenvector of $A + 2B$ corresponding to eigenvalue 4λ.

Learning Goals: Eigenvalues and eigenvectors, properties of eigenvalues and eigenvectors (week 11).
Explanation:

(a) The statement is wrong. Let $A = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}, B = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$. Then A and B both have eigenvalues 1 and 2 while $A - B$ has eigenvalues of -1 and 1.

(b) The statement is correct. Consider an arbitrary real eigenvalue λ of AB with corresponding eigenvector $\mathbf{v}$. We have $AB\mathbf{v} = \lambda\mathbf{v}$. Assume first, that $B\mathbf{v} = \mathbf{0}$. Then $\lambda = 0$ and BA is not full rank, so 0 is also an eigenvalue of BA. Thus, assume now $B\mathbf{v} \neq \mathbf{0}$. Then the calculation $BA(B\mathbf{v}) = \lambda(B\mathbf{v})$ proves that λ is an eigenvalue of BA as well. By a symmetric argument one can show that any eigenvalue of BA is also an eigenvalue of AB.

(c) The statement is wrong. The matrices A and $A^\top$ have the same set of eigenvalues, but the corresponding eigenvectors can be different. For example, $A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ has eigenvalues $\pm i$. The eigenvector of A that corresponds to the eigenvalue i is $\begin{bmatrix}1 & i\end{bmatrix}^\top$. The eigenvector of $A^\top$ that corresponds to the eigenvalue i is $\begin{bmatrix}i & 1\end{bmatrix}^\top$. Specifically, we have

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix} = \begin{bmatrix}i\\ -1\end{bmatrix} = i\begin{bmatrix}1\\ i\end{bmatrix}$$

and

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}i\\ 1\end{bmatrix} = \begin{bmatrix}-1\\ i\end{bmatrix} = i\begin{bmatrix}i\\ 1\end{bmatrix}.$$

(d) The statement is wrong. We have $A\mathbf{v} = \lambda\mathbf{v}$ and $B\mathbf{v} = 2\lambda\mathbf{v}$, then $(A + 2B)\mathbf{v} = 5\lambda\mathbf{v}$.

4. Let $A \in \mathbb{R}^{2\times 2}$ with $\text{rank}(A) = 1$ and $\text{Tr}(A) = 5$. Which of the following is an eigenvalue of A? ***Note: only one answer is correct.***

(a) 1.

(b) -1.

√ **(c)** 5.

(d) -5.

Learning Goals: properties of eigenvalues and eigenvectors (week 11).
Explanation: Let λ_1 and λ_2 be the eigenvalues of A. By $\text{rank}(A) = 1$ we have $\lambda_1 = 0$. We also know from the lecture that $\text{Tr(A)} = \lambda_1 + \lambda_2$. Hence, $\lambda_2 = 5$.

5. Let $A \in \mathbb{R}^{7\times 5}$ be arbitrary. Let $A^\dagger$ be the pseudoinverse of A. Which of the following statements must be **true**? ***Note: only one answer is correct.***

√ **(a)** $\text{rank}(A) = \text{rank}(A^\dagger)$.

(b) $AA^\dagger = I$.

(c) $\text{rank}(A^\dagger) = 5$.

(d) $A^\dagger A = I$.

Learning Goals: Pseudo-inverse, definition and properties (week 9).
Explanation:

(a) From the lecture, we know that $AA^\dagger$ is the projection matrix for projection on $\mathbf{C}(A)$. In particular, this means that $\mathbf{C}(AA^\dagger) = \mathbf{C}(A)$. By the dimension formula, we also have $\dim(\mathbf{C}(AA^\dagger)) = 7 - \dim(\mathbf{N}(AA^\dagger))$. Moreover, observe that $\mathbf{N}(A^\dagger) \subseteq \mathbf{N}(AA^\dagger)$. Hence, we get

$$\text{rank}(A) = \dim(\mathbf{C}(AA^\dagger)) = 7 - \dim(\mathbf{N}(AA^\dagger)) \leq 7 - \dim(\mathbf{N}(A^\dagger)) = \text{rank}(A^\dagger).$$

With a symmetric argument based on $\mathbf{C}(A^\dagger A) = \mathbf{C}(A^\dagger)$, one can show $\text{rank}(A^\dagger) \leq \text{rank}(A)$.

(b) $AA^\dagger$ is a 7×7 matrix. We know that both A and $A^\dagger$ have rank at most 5 because of their dimensions. The statement $AA^\dagger = I$ contradicts this since I is a 7×7 matrix of rank 7. By multiplying together two matrices of rank at most 5, one can only get matrices of rank at most 5.

(c) A could have rank 4. Since a) is correct, this implies $\text{rank}(A^\dagger) = 4$.

(d) $A^\dagger A$ is a 5×5 matrix. Assume that A has rank at most 4 (this is of course possible). The matrix I is a 5×5 matrix of rank 5. But the matrix multiplication $A^\dagger A$ cannot result in a matrix of rank 5 (such as I) under the assumption that the rank of A is at most 4.

6. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -1 & 2 \\ -3 & -2 & 1 \\ 3 & -2 & 1 \end{bmatrix}$. Which of the following choices of matrices U, Σ, V yields a valid singular value decomposition $A = U\Sigma V^\top$? ***Note: only one answer is correct.***

√ **(a)** $U = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$

(b) $U = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3\sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$

(c) $U = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}.$

(d) $U = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -3\sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$

Learning Goals: singular value decomposition (week 13).
Explanation: For **a)**, **b)** and **c)**, all the matrices are such that $A = U\Sigma V^\top$. For **d)**, the matrix product is not defined, the dimensions of matrices do not correspond to allow matrix-multiplication. For U, Σ, V to be matrices constituting the singular value decomposition, U and V need to be orthogonal and Σ needs to only have non-negative values on its diagonal. In **b)**, Σ has negative values and in **c)**, U and V are not orthogonal. **a)** respects all the conditions and is therefore a singular value decomposition of A.