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From the Farkas lemma to LP duality

The point of departure

This material can be found in any textbook about linear optimization, see for example
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Theorem (The Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Either there exists a vector $x \in \mathbb{R}^n$ such that $Ax \leq b$ or there exists a vector $y \in \mathbb{R}^m$ such that $y \geq 0, y^T A = 0$ and $y^T b < 0$.

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Theorem (LP Duality)

(van Neumann 1947, Gale, Kuhn, Tucker 1951)
Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Assume that $P = \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ and $D = \{y \in \mathbb{R}^m \mid y^T A = c^T,\ y \geq 0\} \neq \emptyset$. Then

$$\max\{c^T x \mid x \in P\} = \min\{y^T b \mid y \in D\}$$

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- Since $D \neq \emptyset$ and $P \neq \emptyset$, pick any $\hat{x} \in P$ and $\hat{y} \in D$. This gives

$$c^T \hat{x} = \hat{y}^T A \hat{x} \leq \hat{y}^T b, \tag{1}$$

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- Let us assume the following fact.
- Since the optimal value is $\delta = \min\{y^T b \mid y \in D\}$ is bounded, it is attained, i.e., there exists an optimal solution $y^* \in D$ such that $\delta = b^T y^*$.

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- Suppose that $\lambda = 0$. Then we obtain $z^T A = 0$, $z^T b < 0$, $z \geq 0$ which by the Farkas lemma gives $P = \emptyset$. This contradicts our assumptions.
- Hence $\lambda > 0$. Then define the vector $y = \frac{1}{\lambda} z$. Now Equation (2) reads

$$y^T A = c^T,\ y^T b < \delta, y \geq 0.$$

This system has a solution. But this contradicts that $\delta = \min\{y^T b \mid y \in D\}$.