

# Lineare Algebra

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Gauss Elimination, Worked Example

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## Solve a system of equations with Gauss elimination and back substitution

System of linear equations

$$\begin{array}{ccccrc} & 3x_2 & -1x_3 & +2x_4 & = & -1 \\ 2x_1 & +1x_2 & -1x_3 & +2x_4 & = & 5 \\ -2x_1 & +5x_2 & -2x_3 & +6x_4 & = & 4 \\ 4x_1 & +11x_2 & -4x_3 & +8x_4 & = & 2 \end{array}$$

Matrix vector form  $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 0 & 3 & -1 & 2 \\ 2 & 1 & -1 & 2 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \end{bmatrix}}_{\mathbf{b}}$$

Algorithm:

- ▶ Turn the red entries into 0 using row subtractions and row exchanges!
- ▶ Use back substitution to solve for  $\mathbf{x}$ !

## Zero pivot: Exchange rows 1 and 2

$$\begin{array}{ccc} & \mathbf{A} & \mathbf{b} \\ \left[ \begin{array}{cccc} \mathbf{0} & 3 & -1 & 2 \\ \mathbf{2} & 1 & -1 & 2 \\ -2 & \mathbf{5} & -2 & 6 \\ \mathbf{4} & \mathbf{11} & -4 & 8 \end{array} \right] & \left[ \begin{array}{c} -1 \\ 5 \\ 4 \\ 2 \end{array} \right] & \begin{array}{l} \leftarrow \text{row 1} \\ \leftarrow \text{row 2} \end{array} \\ \\ & P_{12} \cdot \downarrow & P_{12} \cdot \downarrow \\ \left[ \begin{array}{cccc} \mathbf{2} & 1 & -1 & 2 \\ \mathbf{0} & 3 & -1 & 2 \\ -2 & \mathbf{5} & -2 & 6 \\ \mathbf{4} & \mathbf{11} & -4 & 8 \end{array} \right] & \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] & \end{array}$$

permutation matrix:  $P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## Elimination in column 1: Subtract $0 \cdot$ row 1 from row 2

$$\begin{array}{c}
 \begin{array}{cc}
 & \mathbf{A} \\
 \left[ \begin{array}{cccc}
 \mathbf{2} & 1 & -1 & 2 \\
 \mathbf{0} & 3 & -1 & 2 \\
 -2 & \mathbf{5} & -2 & 6 \\
 4 & \mathbf{11} & -4 & 8
 \end{array} \right] & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}
 \end{array} \\
 \\
 \begin{array}{c}
 \begin{array}{cc}
 & \mathbf{A} \\
 \left[ \begin{array}{cccc}
 \mathbf{2} & 1 & -1 & 2 \\
 \mathbf{0} & 3 & -1 & 2 \\
 -2 & \mathbf{5} & -2 & 6 \\
 4 & \mathbf{11} & -4 & 8
 \end{array} \right] & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}
 \end{array} \\
 \\
 \begin{array}{c}
 \begin{array}{cc}
 & \mathbf{A} \\
 \left[ \begin{array}{cccc}
 \mathbf{2} & 1 & -1 & 2 \\
 \mathbf{0} & 3 & -1 & 2 \\
 -2 & \mathbf{5} & -2 & 6 \\
 4 & \mathbf{11} & -4 & 8
 \end{array} \right] & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}
 \end{array}
 \end{array}
 \end{array}
 \end{array}
 \begin{array}{l}
 \\
 \text{row 2: } \mathbf{0} \quad 3 \quad -1 \quad 2 \quad -1 \\
 \\
 \begin{array}{l}
 0 \cdot \text{row 1: } \frac{0}{0} \quad \frac{0}{3} \quad \frac{0}{-1} \quad \frac{0}{2} \quad \frac{0}{-1} \\
 = \text{row 2': } \mathbf{0} \quad 3 \quad -1 \quad 2 \quad -1
 \end{array}
 \end{array}
 \end{array}$$

elimination matrix:  $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Elimination in column 1: Subtract  $(-1) \cdot$  row 1 from row 3

$$\begin{array}{c} A \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ -2 & 5 & -2 & 6 \\ 4 & 11 & -4 & 8 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 4 \\ 2 \end{array} \right] \end{array}$$

$$\text{row 3: } \quad -2 \quad 5 \quad -2 \quad 6 \quad 4$$

$$\begin{array}{c} E_{31} \cdot \downarrow \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 6 & -3 & 8 \\ 4 & 11 & -4 & 8 \end{array} \right] \end{array} \quad \begin{array}{c} E_{31} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ 2 \end{array} \right] \end{array}$$

$$\frac{(-1) \cdot \text{row 1:}}{\quad} \quad \frac{-2}{\quad} \quad \frac{-1}{\quad} \quad \frac{1}{\quad} \quad \frac{-2}{\quad} \quad \frac{-5}{\quad}$$

$$= \text{row 3':} \quad 0 \quad 6 \quad -3 \quad 8 \quad 9$$

$$\text{elimination matrix: } E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Elimination in column 1: Subtract $2 \cdot$ row 1 from row 4

$$\begin{array}{c}
 \begin{array}{cc}
 & \mathbf{A} \\
 \left[ \begin{array}{cccc}
 \mathbf{2} & 1 & -1 & 2 \\
 0 & 3 & -1 & 2 \\
 0 & \mathbf{6} & -3 & 8 \\
 \mathbf{4} & \mathbf{11} & -4 & 8
 \end{array} \right] & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ 2 \end{array} \right] \end{array}
 \end{array} \\
 \\
 \begin{array}{cc}
 E_{41} \cdot \downarrow & \\
 \left[ \begin{array}{cccc}
 \mathbf{2} & 1 & -1 & 2 \\
 0 & 3 & -1 & 2 \\
 0 & \mathbf{6} & -3 & 8 \\
 0 & \mathbf{9} & -2 & 4
 \end{array} \right] & \begin{array}{c} E_{41} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ -8 \end{array} \right] \end{array}
 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{row 4 : } \mathbf{4} \quad \mathbf{11} \quad -4 \quad 8 \quad 2 \\
 \\
 \hline
 2 \cdot \text{row 1 : } \quad 4 \quad 2 \quad -2 \quad 4 \quad 10 \\
 \\
 \hline
 = \text{row 4}' : \quad 0 \quad 9 \quad -2 \quad 4 \quad -8
 \end{array}$$

elimination matrix:  $E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}$

## Elimination in column 2: Subtract $2 \cdot$ row 2 from row 3

$$\begin{array}{c} \mathbf{A} \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & \mathbf{6} & -3 & 8 \\ 0 & \mathbf{9} & -2 & 4 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 9 \\ -8 \end{array} \right] \end{array}$$

$$\text{row 3 : } \quad 0 \quad \mathbf{6} \quad -3 \quad 8 \quad 9$$

$$\begin{array}{c} E_{32} \cdot \downarrow \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & \mathbf{0} & -1 & 4 \\ 0 & \mathbf{9} & -2 & 4 \end{array} \right] \end{array} \quad \begin{array}{c} E_{32} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -8 \end{array} \right] \end{array}$$

$$\begin{array}{l} 2 \cdot \text{row 2 :} \\ \hline = \text{row 3' :} \end{array} \quad \begin{array}{cccccc} 0 & 6 & -2 & 4 & -2 \\ \hline 0 & 0 & -1 & 4 & 11 \end{array}$$

$$\text{elimination matrix: } E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

## Elimination in column 2: Subtract $3 \cdot$ row 2 from row 4

$$\begin{array}{cc}
 \begin{array}{c} A \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & \mathbf{9} & -2 & 4 \end{array} \right] \end{array} & \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -8 \end{array} \right] \end{array} \\
 \\
 \begin{array}{c} E_{42} \cdot \downarrow \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & \mathbf{3} & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right] \end{array} & \begin{array}{c} E_{42} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -5 \end{array} \right] \end{array}
 \end{array}
 \quad
 \begin{array}{l}
 \text{row 4 : } \quad 0 \quad \mathbf{9} \quad -2 \quad 4 \quad -8 \\
 \\
 \begin{array}{r}
 3 \cdot \text{row 2 : } \quad 0 \quad 9 \quad -3 \quad 6 \quad -3 \\
 \hline
 = \text{row 4}' : \quad 0 \quad 0 \quad \mathbf{1} \quad -2 \quad -5
 \end{array}
 \end{array}$$

elimination matrix:  $E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}$



Elimination in column 3: Subtract  $(-1) \cdot$  row 3 from row 4

$$\begin{array}{c} A \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & \mathbf{1} & -2 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ -5 \end{array} \right] \end{array}$$

$$\text{row 4 : } \quad 0 \quad 0 \quad \mathbf{1} \quad -2 \quad -5$$

$$\begin{array}{c} E_{43} \cdot \downarrow \\ \left[ \begin{array}{cccc} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{array} \right] \end{array} \quad \begin{array}{c} E_{43} \cdot \downarrow \\ \left[ \begin{array}{c} 5 \\ -1 \\ 11 \\ 6 \end{array} \right] \end{array}$$

$$\begin{array}{l} (-1) \cdot \text{row 3 : } \quad 0 \quad 0 \quad 1 \quad -4 \quad -11 \\ \hline = \text{row 4}' : \quad 0 \quad 0 \quad 0 \quad -2 \quad -6 \end{array}$$

elimination matrix:  $E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

## Back substitution to compute $x_4, x_3, x_2, x_1$

Matrix vector form  $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 5 \\ -1 \\ 11 \\ 6 \end{bmatrix}}_{\mathbf{b}}$$

System of linear equations

$$\begin{aligned} 2x_1 + 1x_2 - 1x_3 + 2x_4 &= 5 \\ & 3x_2 - 1x_3 + 2x_4 = -1 \\ & & -1x_3 + 4x_4 = 11 \\ & & & 2x_4 = 6 \end{aligned}$$

eq.	before substitution	after substitution	solution	
1	$2x_1 + 1x_2 - 1x_3 + 2x_4 = 5$	$2x_1 + 3 = 5$	$x_1 = 1$	↑
2	$3x_2 - 1x_3 + 2x_4 = -1$	$3x_2 + 5 = -1$	$x_2 = -2$	
3	$-1x_3 + 4x_4 = 11$	$-1x_3 + 12 = 11$	$x_3 = 1$	
4	$2x_4 = 6$		$x_4 = 3$	