

# Lineare Algebra

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Gauss Elimination, Worked Example

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## Solve a system of equations with Gauss elimination and back substitution

System of linear equations

$$\begin{array}{rccccrcr} 1x_1 & +9x_2 & +2x_3 & +5x_4 & = & -2 \\ 7x_1 & -3x_2 & +2x_3 & & = & 8 \\ 6x_1 & +8x_2 & +1x_3 & -1x_4 & = & 2 \\ -2x_1 & +3x_2 & & -5x_4 & = & 3 \end{array}$$

Matrix vector form  $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 1 & 9 & 2 & 5 \\ 7 & -3 & 2 & 0 \\ 6 & 8 & 1 & -1 \\ -2 & 3 & 0 & -5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 8 \\ 2 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

Algorithm:

- ▶ Turn the red entries into 0 using row subtractions and row exchanges!
- ▶ Use back substitution to solve for  $\mathbf{x}$ !

## Elimination in column 1: Subtract $7 \cdot$ row 1 from row 2

$$\begin{array}{c} A \\ \left[ \begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{7} & -3 & 2 & 0 \\ \mathbf{6} & \mathbf{8} & 1 & -1 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} -2 \\ 8 \\ 2 \\ 3 \end{array} \right] \end{array}$$

$$\text{row 2: } \mathbf{7} \quad -3 \quad 2 \quad 0 \quad 8$$

$$\begin{array}{c} E_{21} \cdot \downarrow \\ \left[ \begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ \mathbf{0} & -66 & -12 & -35 \\ \mathbf{6} & \mathbf{8} & 1 & -1 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] \end{array} \quad \begin{array}{c} E_{21} \cdot \downarrow \\ \left[ \begin{array}{c} -2 \\ 22 \\ 2 \\ 3 \end{array} \right] \end{array}$$

$$\begin{array}{l} 7 \cdot \text{row 1:} \\ \hline = \text{row 2':} \end{array} \quad \begin{array}{c} 7 \\ \mathbf{0} \end{array} \quad \begin{array}{c} 63 \\ -66 \end{array} \quad \begin{array}{c} 14 \\ -12 \end{array} \quad \begin{array}{c} 35 \\ -35 \end{array} \quad \begin{array}{c} -14 \\ 22 \end{array}$$

elimination matrix:  $E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## Elimination in column 1: Subtract 6 · row 1 from row 3

$$\begin{array}{c} \mathbf{A} \\ \left[ \begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ \mathbf{6} & \mathbf{8} & 1 & -1 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} -2 \\ 22 \\ 2 \\ 3 \end{array} \right] \end{array}$$

$$\begin{array}{c} E_{31} \cdot \downarrow \\ \left[ \begin{array}{cccc} \mathbf{1} & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & -46 & -11 & -31 \\ -2 & \mathbf{3} & \mathbf{0} & -5 \end{array} \right] \end{array} \quad \begin{array}{c} E_{31} \cdot \downarrow \\ \left[ \begin{array}{c} -2 \\ 22 \\ 14 \\ 3 \end{array} \right] \end{array}$$

$$\text{row 3: } \mathbf{6} \quad \mathbf{8} \quad 1 \quad -1 \quad 2$$

$$\frac{6 \cdot \text{row 1:}}{\quad} \quad \frac{6}{\quad} \quad \frac{54}{\quad} \quad \frac{12}{\quad} \quad \frac{30}{\quad} \quad \frac{-12}{\quad}$$

$$= \text{row 3': } \quad 0 \quad -46 \quad -11 \quad -31 \quad 14$$

$$\text{elimination matrix: } E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Elimination in column 2: Subtract  $\frac{23}{33} \cdot$  row 2 from row 3

$$\begin{array}{c} A \\ \left[ \begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & -46 & -11 & -31 \\ 0 & 21 & 4 & 5 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} -2 \\ 22 \\ 14 \\ -1 \end{array} \right] \end{array}$$

$$\text{row 3 : } 0 \quad -46 \quad -11 \quad -31 \quad 14$$

$$\begin{array}{c} E_{32} \cdot \downarrow \\ \left[ \begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 21 & 4 & 5 \end{array} \right] \end{array} \quad \begin{array}{c} E_{32} \cdot \downarrow \\ \left[ \begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ -1 \end{array} \right] \end{array}$$

$$\begin{array}{l} \frac{23}{33} \cdot \text{row 2 : } 0 \quad -46 \quad -\frac{92}{11} \quad -\frac{805}{33} \quad \frac{46}{3} \\ \hline = \text{row 3' : } 0 \quad 0 \quad -\frac{29}{11} \quad -\frac{218}{33} \quad -\frac{4}{3} \end{array}$$

$$\text{elimination matrix: } E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{23}{33} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elimination in column 2: Subtract  $(-\frac{7}{22}) \cdot \text{row 2}$  from row 4

$$\begin{array}{c} \mathbf{A} \\ \left[ \begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 21 & 4 & 5 \end{array} \right] \end{array} \quad \begin{array}{c} \mathbf{b} \\ \left[ \begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ -1 \end{array} \right] \end{array}$$

$$\text{row 4 : } \quad 0 \quad 21 \quad 4 \quad 5 \quad -1$$

$$\begin{array}{c} E_{42} \cdot \downarrow \\ \left[ \begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & \frac{2}{11} & -\frac{135}{22} \end{array} \right] \end{array} \quad \begin{array}{c} E_{42} \cdot \downarrow \\ \left[ \begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ 6 \end{array} \right] \end{array}$$

$$\frac{(-\frac{7}{22}) \cdot \text{row 2 :}}{\quad} \quad \frac{0}{\quad} \quad \frac{21}{\quad} \quad \frac{\frac{42}{11}}{\quad} \quad \frac{\frac{245}{22}}{\quad} \quad \frac{-7}{\quad}$$

$$= \text{row 4' : } \quad 0 \quad 0 \quad \frac{2}{11} \quad -\frac{135}{22} \quad 6$$

elimination matrix:  $E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{7}{22} & 0 & 1 \end{bmatrix}$

Elimination in column 3: Subtract  $(-\frac{2}{29}) \cdot \text{row 3}$  from row 4

$$\begin{array}{c}
 \mathbf{A} \qquad \qquad \mathbf{b} \\
 \left[ \begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & \frac{2}{11} & -\frac{135}{22} \end{array} \right] \quad \left[ \begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ 6 \end{array} \right] \\
 \text{row 4 :} \quad 0 \quad 0 \quad \frac{2}{11} \quad -\frac{135}{22} \quad 6 \\
 E_{43} \cdot \downarrow \qquad E_{43} \cdot \downarrow \\
 \left[ \begin{array}{cccc} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & 0 & -\frac{1147}{174} \end{array} \right] \quad \left[ \begin{array}{c} -2 \\ 22 \\ -\frac{4}{3} \\ \frac{514}{87} \end{array} \right] \\
 \frac{(-\frac{2}{29}) \cdot \text{row 3} : \quad 0 \quad 0 \quad \frac{2}{11} \quad -\frac{436}{957} \quad -\frac{8}{87}}{= \text{row 4}' : \quad 0 \quad 0 \quad 0 \quad -\frac{1147}{174} \quad \frac{514}{87}}
 \end{array}$$

elimination matrix:  $E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{29} & 1 \end{bmatrix}$



# Back substitution to compute $x_4, x_3, x_2, x_1$

Matrix vector form  $A\mathbf{x} = \mathbf{b}$

$$\underbrace{\begin{bmatrix} 1 & 9 & 2 & 5 \\ 0 & -66 & -12 & -35 \\ 0 & 0 & -\frac{29}{11} & -\frac{218}{33} \\ 0 & 0 & 0 & -\frac{1147}{174} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} -2 \\ 22 \\ -\frac{4}{3} \\ \frac{514}{87} \end{bmatrix}}_b$$

System of linear equations

$$\begin{aligned} 1x_1 + 9x_2 + 2x_3 + 5x_4 &= -2 \\ -66x_2 - 12x_3 - 35x_4 &= 22 \\ -\frac{29}{11}x_3 - \frac{218}{33}x_4 &= -\frac{4}{3} \\ -\frac{1147}{174}x_4 &= \frac{514}{87} \end{aligned}$$

eq.	before substitution	after substitution	solution
1	$1x_1 + 9x_2 + 2x_3 + 5x_4 = -2$	$1x_1 - \frac{2527}{1147} = -2$	$x_1 = \frac{233}{1147}$
2	$-66x_2 - 12x_3 - 35x_4 = 22$	$-66x_2 - \frac{1892}{1147} = 22$	$x_2 = -\frac{411}{1147}$
3	$-\frac{29}{11}x_3 - \frac{218}{33}x_4 = -\frac{4}{3}$	$-\frac{29}{11}x_3 + \frac{224104}{37851} = -\frac{4}{3}$	$x_3 = \frac{3156}{1147}$
4	$-\frac{1147}{174}x_4 = \frac{514}{87}$		$x_4 = -\frac{1028}{1147}$

