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Lecture 13: Bases and dimension

The target

Understand generators of a vector space: Key concept is this of a basis: this is a subset of linearly independent elements in the vector space whose span is the entire vector space.

Definition

Let V be a vector space. Let $G \subseteq V$ be a (possibly infinite) subset of vectors. A linear combination of G is a sum of the form

$$\sum_{v \in F} \lambda_v v,$$

where $F \subseteq G$ is a finite subset of G and $\lambda_v \in \mathbb{R}$ for all $v \in F$.

Lemma

Let V be a vector space. Let $G \subseteq V$. Every linear combination of G is again an element in V.

Linear combinations

Proof of the lemma

"Every linear combination of $G \subseteq V$ is again an element in V."

- Let $F = \{v_1, \ldots, v_n\} \subseteq G$ be a finite subset of G. Consider the linear combination
$$\sum_{j=1}^{n} \lambda_j v_j.$$
- By definition of a vector space, $w_j := \lambda_j v_j \in V$ for all $j = 1, \ldots, n$.
- Accordingly, $w_1 + w_2 \in V$.
- Use an inductive argument:
- If $w := w_1 + \ldots + w_{j-1} \in V$ then,
$$w_1 + w_2 + \cdots + w_{j-1} + w_j = w + w_j \in V.$$

The concept of a basis

Why not infinite linear combinations?

The previous lemma may fail. Let V be the vector space of polynomials with real coefficients in one variable x. Then $1, x, x^2, x^3, \ldots \in V$, but

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \notin V.$$

Definition

Let V be a vector space. Let $G \subseteq V$ be a subset of elements in V.

- $$Span(G) = \{ \sum_{v \in F} \lambda_v v \mid \lambda_v \in \mathbb{R} \text{ for all } v \in F,\ F \subseteq G \text{ finite}\}.$$
- G is linearly independent if no vector $v \in G$ is a linear combination of $G \setminus \{v\}$.
- G is linearly dependent if there exists a vector $w \in G$ that is a linear combination of $G \setminus \{w\}$.
- $B \subseteq V$ is a basis of V if B is linearly independent and $Span(B) = V$.

Two observations

Let V be a vect. sp. $G \subseteq V$ is linearly independent if and only if for all $F \subseteq G$, F finite, $\sum_{v \in F} \lambda_v v = 0$ implies $\lambda_v = 0$ for all $v \in F$.

- Suppose F is finite and $\sum_{v \in F} \lambda_v v = 0$ where $\lambda_w \neq 0$ for $w \in F$.
- Then $\lambda_w w = -\sum_{v \in F \setminus \{w\}} \lambda_v v \iff w = \frac{-1}{\lambda_w} \sum_{v \in F \setminus \{w\}} \lambda_v v$ and hence,
- $w \in G$ is a linear combination of $G \setminus \{w\}$.
- Conversely, if $w = \sum_{v \in F} \lambda_v v$ where $F \subseteq G$ finite and $w \in G$, then we obtain
- $0 = \sum_{v \in F} \lambda_v v - 1 \cdot w$.

Every set B of m linearly ind. vectors in $\mathbb{R}^m$ is a basis of $\mathbb{R}^m$.

- Let $B = \{v_1, v_2, \ldots, v_m\}$. We need to show that $Span(B) = \mathbb{R}^m$!
- Define the matrix $A \in \mathbb{R}^{m \times m}$ with columns $v_1, v_2, \ldots, v_m$.
- From Theorem 3.11 it follows that for every $v \in \mathbb{R}^m$, $Ax = v$ has a unique solution $x \in \mathbb{R}^m$ and hence,

$$v = \sum_{j=1}^{m} x_j v_j$$

Steinitz exchange lemma

Lemma.

Let V be a vector space, $F \subseteq V$ finite and linearly independent, $G \subseteq V$ finite with $Span(G) = V$. Then

(i) $|F| \leq |G|$.

(ii) There exists a subset $E \subseteq G$ of size $|G| - |F|$ such that $Span(F \cup E) = V$.

Proof by induction on $f = |F|$.

- if $f = 0$, then $F = \emptyset$. Hence, (i) is clear and for (ii), take $E = G$.
- Let $f > 0$. Suppose the statement is correct for all numbers smaller than f.
- Choose $u \in F$, $F' = F \setminus \{u\}$, $g = |G|$. From the induction hypothesis applied to F' we have
  (i) $g \geq f - 1$.
  (ii) There exists a subset $E' \subseteq G$ of size $g - (f-1)$ with $Span(F' \cup E') = V$.

Proof of the Steinitz exchange lemma continued

- $u \in Span(F' \cup E')$. Since F is lin. ind., $u \notin Span(F')$. This gives
$$u = \sum_{v \in F' \cup E'} \lambda_v v, \text{ where } \lambda_w \neq 0 \text{ for some } w \in E'.$$
- Hence $|E'| = g - (f-1) \geq 1 \iff g \geq f$ and we showed statement (i).

To show (ii), let $E := E' \setminus \{w\}$.

- We have $F' = F \setminus \{u\}$ and $span(F' \cup E') = V$.
- u is a linear combination of $F' \cup E' = F' \cup E \cup \{w\}$,
$$u = \sum_{v \in F' \cup E} \lambda_v v + \lambda_w w \iff w = \frac{1}{\lambda_w}(u - \sum_{v \in F' \cup E} \lambda_v v).$$
- Hence w is a linear combination of $\{u\} \cup F' \cup E = F \cup E$.
- This gives $Span(F \cup E) = Span(\{u\} \cup F' \cup E)$.
$$Span(\{u\} \cup F' \cup E) = Span(\{u\} \cup \{w\} \cup F' \cup E) = Span(\{u\} \cup F' \cup E') = V.$$

Finitely generated vector spaces I

Definition

A vector space V is called finitely generated if there exists a finite subset $G \subseteq V$ with $Span(G) = V$.

$\mathbb{R}^m$ is finitely generated, whereas $\mathbb{R}[x]$ is not

Theorem 4.22

Let V be a finitely generated vector space, let $G \subseteq V$ be a finite subset of V with $Span(G) = V$. Then V has a basis $B \subseteq G$.

Proof.

- If G is linearly independent, then $B = G$ is a basis by definition.
- Otherwise, there exists $v \in G$ that is a linear combination of the other vectors. We have that $Span(G \setminus \{v\}) = Span(G) = V$.
- Replace G with $G \setminus \{v\}$ and iterate.
- Since V is finitely generated this process finally stops with $B = G$.

Finitely generated vector spaces II

Theorem 4.20

Let V be a finitely generated vector space. Let $B, B' \subseteq V$ be two finite bases of V. Then, $|B| = |B'|$.

Proof.

- Bases B and B' are linearly independent and $Span(B) = Span(B') = V$.
- Apply Steinitz exchange lemma (i):
- $F = B, G = B' \Rightarrow |B| \leq |B'|$.
- $F = B', G = B \Rightarrow |B'| \leq |B|$.

Theorem 4.20 and Theorem 4.22

Let V be a finitely generated vector space. V has a finite basis $B \subseteq V$ and whenever $B, B' \subseteq V$ are two finite bases of V, then $|B| = |B'|$.

Dimension of a vector space

Definition

Let V be a finitely generated vector space. Then $\dim(V)$, the dimension of V, is the size of any basis B of V. (Note that $\dim(\mathbb{R}^m) = m$)

Lemma 4.24

Let V be a vector space of dimension d.

(i) Let $F \subseteq V$ be a set of d linearly independent vectors. F is a basis of V.

(ii) Let $G \subseteq V$ be a set of d vectors with $Span(G) = V$. G is a basis of V.

Proof.

- Let G be a basis of V. Steinitz exchange Lemma applies with F and G.
- $|F| = |G| = d \Rightarrow E = \emptyset$. $Span(F) = Span(F \cup E) = V \Rightarrow F$ is a basis.
- From Thm 4.22, there exists a basis $B \subseteq G$ of size d, i.e., $|B| = |G|$.

$$\Rightarrow \quad B = G \quad \Rightarrow \quad G \text{ is a basis.}$$