Linear Algebra
ETH Zürich, HS 2023, 401-0131-00L

The Computer Science Lens

Lights Out!

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$\mathbb{F}$-vector spaces – where $\mathbb{F}$ is a *field* ($\mathbb{R}$ is only one of many fields)

A $\mathbb{F}$-vector space[1] is a triple $(V, +, \cdot)$ where V is a set (the vectors), and

$$
\begin{array}{lcll}
+ & : & V \times V \to V & \text{a function (vector addition),} \\
\cdot & : & \mathbb{F} \times V \to V & \text{a function (scalar multiplication),}
\end{array}
$$

satisfying the following *axioms* (rules) for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all $c, d \in \mathbb{F}$.

1.	$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	commutativity
2.	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	associativity
3.	There is a vector $\mathbf{0}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v}$	zero vector
4.	There is a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	negative vector
5.	$1 \cdot \mathbf{v} = \mathbf{v}$	identity element
6.	$(c \cdot d)\mathbf{v} = c \cdot (d \cdot \mathbf{v})$	compatibility
7.	$c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$	distributivity over $+$
8.	$(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$	distributivity over $+$ in $\mathbb{F}$

[1] "real" stands for real numbers $c \in \mathbb{R}$ as scalars

Fields

A field is a triple $(F, +, \cdot)$ where F is a set (the numbers), and

$$
\begin{array}{lcll}
+ & : & F \times F \to F & \text{a function (addition of two numbers),} \\
\cdot & : & F \times F \to F & \text{a function (multiplication of two numbers),}
\end{array}
$$

satisfying the following *axioms* (rules) for all $a, b, c \in \mathbb{F}$:

don't learn them by heart!

1.	$a + b = b + a$	commutativity of $+$
2.	$a \cdot b = b \cdot a$	commutativity of $\cdot$
3.	$a + (b + c) = (a + b) + c$	associativity of $+$
4.	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$	associativity of $\cdot$
5.	there is a number 0 such that $a + 0 = a$ for all a	zero
6.	there is a number $1 \neq 0$ such that $a \cdot 1 = a$ for all a	one
7.	There is a number $-a$ such that $a + (-a) = 0$	negative
8.	If $a \neq 0$, there is a number a^{-1} such that $a \cdot a^{-1} = 1$	inverse
9.	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	distributivity

Examples of fields

- $\mathbb{R}$ (real numbers)
- $\mathbb{C}$ (complex numbers)
- $\mathbb{Q}$ (rational numbers)

Non-examples:

- $\mathbb{Z}$ (integers): no inverses
- $\mathbb{N}$ (natural numbers): no negatives

Finite fields of prime order (very important in cryptography):

- $\mathbb{F}_p = (\{0, 1, \ldots, p-1\}, +, \cdot)$, where p is a prime number.

$$a + b = (\underbrace{a + b}_{+ \text{ in } \mathbb{N}}) \bmod p \qquad a \cdot b = (\underbrace{a \cdot b}_{\cdot \text{ in } \mathbb{N}}) \bmod p$$

- $p = 2 : \mathbb{F}_2 = (\{0, 1\}, +, \cdot)$. The *smallest possible* field (every field has 0 and 1).

$(a + b) \bmod 2$:

+	0	1
0	0	1
1	1	0

$(a \cdot b) \bmod 2$:

·	0	1
0	0	0
1	0	1

In all cases, the field axioms have been checked.

The field $\mathbb{F}_2$: Calculating with bits (value 0 or 1)

Adding two bits: the logical exclusive or

+	0	1
0	0	1
1	1	0

$$b_1 + b_2 = \begin{cases} 1 & \text{if either } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad = b_1 \text{ XOR } b_2$$

Multiplying two bits: the logical and

·	0	1
0	0	0
1	0	1

$$b_2 \cdot b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ and } b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad = b_1 \text{ AND } b_2$$

Adding more bits:

$$b_1 + b_2 + \cdots + b_n = \begin{cases} 1 & \text{if an odd number of } b_i\text{'s is 1} \\ 0 & \text{if an even number of } b_i\text{'s is 1} \end{cases}$$

$$\begin{array}{rcl} & & 3 \bmod 2 \\ & & \parallel \\ 0+1+1+0+1 & = & 1 \\ 1+0+1+1+1 & = & 0 \\ & & \parallel \\ & & 4 \bmod 2 \end{array}$$

For every field $\mathbb{F}$, we have the $\mathbb{F}$-vector space $\mathbb{F}^n$ (if $\mathbb{F} = \mathbb{R}$, this is $\mathbb{R}^n$)

Vectors: $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, where $v_1, v_2, \ldots, v_n \in \mathbb{F}$.

Vector addition:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}, \quad \text{where } + \text{ is the addition in } \mathbb{F}$$

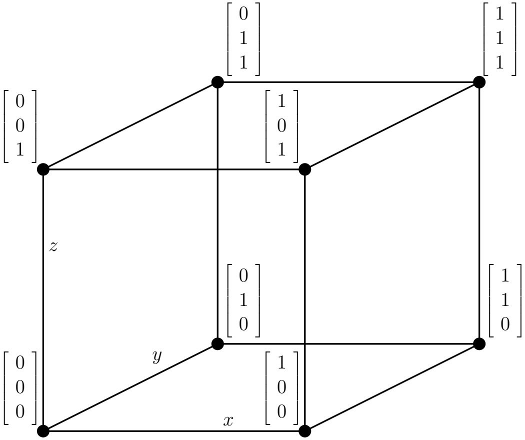
Scalar multiplication:

$$c \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot tv_n \end{bmatrix}, \quad \text{where } \cdot \text{ is the multiplication in } \mathbb{F}$$

Bit vectors: elements of the vector space $\mathbb{F}_2^n$

$\mathbb{F}_2^n$ contains 2^n vectors.

$n = 3$:



"Hamming cube"

Linear combinations in $\mathbb{F}_2^n$

$$\lambda_1 \mathbf{v}_1 + \cdots + \underset{\substack{\downarrow \\ 1 \,:\, \text{take } \mathbf{v}_i \\ 0 \,:\, \text{don't take } \mathbf{v}_i}}{\lambda_i \mathbf{v}_i} + \cdots + \lambda_n \mathbf{v}_n$$

Combinations are just sums of vectors (the ones we take).

Vectors are independent if we can only get $\mathbf{0}$ by taking none of them.

$$\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}}_{\text{independent}} \qquad \underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}}_{\text{dependent}} : \quad \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

In $\mathbb{R}^3$, these three vectors would be independent!

Systems of linear equations in $\mathbb{F}^n$

Everything we do in $\mathbb{R}^n$ works the same way in $\mathbb{F}^n$:

- Matrices
- $A\mathbf{x} = \mathbf{b}$ and Gauss elimination
- Inverse matrices
- Gauss-Jordan elimination
- Full solution of $A\mathbf{x} = \mathbf{b}$ (Week 7)
- ...

Example ($\mathbb{F}_2^5$): solve for the bit vector $\mathbf{x}$!

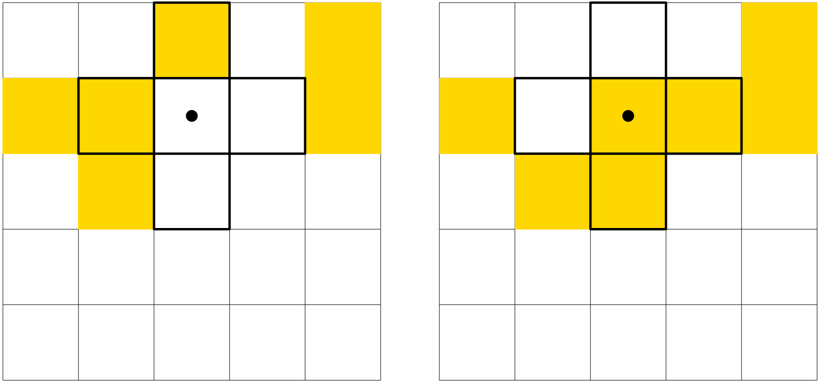
$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Take columns $1, 3, 5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Application: Game "Lights out!"

$n \times n$ grid of buttons (original game: 5×5), some are on (yellow):

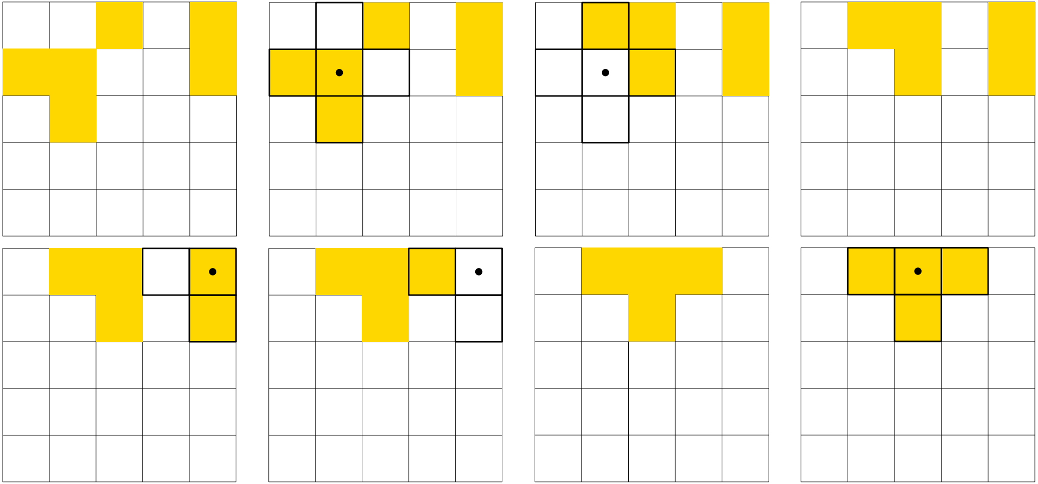


Pressing a button...switches it (on $\leftrightarrow$ off) and all its neighbors.

Goal: Repeatedly press buttons until all are off!

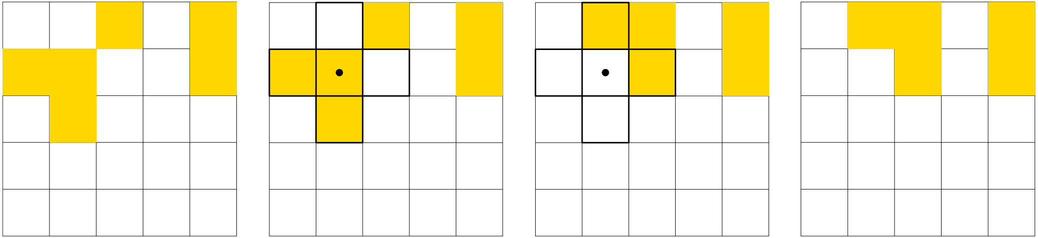
Lights Out!

Solution



Done after this button!

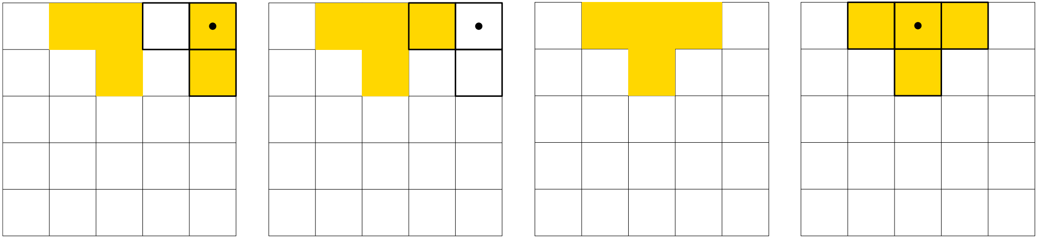
First solution step, mathematically



$$\begin{array}{|c|c|c|c|c|}\hline 0&0&1&0&1\\\hline 1&1&0&0&1\\\hline 0&1&0&0&0\\\hline 0&0&0&0&0\\\hline 0&0&0&0&0\\\hline\end{array} \quad + \quad \begin{array}{|c|c|c|c|c|}\hline 0&1&0&0&0\\\hline 1&1&1&0&0\\\hline 0&1&0&0&0\\\hline 0&0&0&0&0\\\hline 0&0&0&0&0\\\hline\end{array} \quad = \quad \begin{array}{|c|c|c|c|c|}\hline 0&1&1&0&1\\\hline 0&0&1&0&1\\\hline 0&0&0&0&0\\\hline 0&0&0&0&0\\\hline 0&0&0&0&0\\\hline\end{array}$$

vector in $\mathbb{F}_2^{25}$ — "button vector" $\mathbf{b}_7$ in $\mathbb{F}_2^{25}$ — vector in $\mathbb{F}_2^{25}$

Second solution step, mathematically



0	1	1	0	1
0	0	1	0	1
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

vector in $\mathbb{F}_2^{25}$

$+$

0	0	0	1	1
0	0	0	0	1
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

"button vector" $\mathbf{b}_5$ in $\mathbb{F}_2^{25}$

$=$

0	1	1	1	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

vector in $\mathbb{F}_2^{25}$

Lights Out, mathematically

Given a vector $\mathbf{v} \in \mathbb{F}_2^{25}$, produce $\mathbf{0} \in \mathbb{F}_2^{25}$ by adding suitable button vectors!

Same problem ("play the game backwards"): starting from $\mathbf{0}$, produce $\mathbf{v}$ by adding suitable button vectors!

0	0	1	0	1
1	1	0	0	1
0	1	0	0	0
0	0	0	0	0
0	0	0	0	0

$\mathbf{v}$

$=$

0	1	0	0	0
1	1	1	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	0

$\mathbf{b}_7$

$+$

0	0	0	1	1
0	0	0	0	1
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$\mathbf{b}_5$

$+$

0	1	1	1	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$\mathbf{b}_3$

No button vector is needed twice ($\mathbf{b}_i + \mathbf{b}_i = \mathbf{0}$, no effect).

Order of button vectors doesn't matter (commutativity)!

Lights Out: A system of linear equations in $\mathbb{F}_2^{25}$!

To win the game with initial configuration $\mathbf{v} \in \mathbb{F}_2^{25}$, solve

$$\mathbf{v} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots x_{25}\mathbf{b}_{25}$$

with all $x_i \in \mathbb{F}_2$ (0 or 1).

This is a system of linear equations with 25 equations in 25 unknowns:

$$\underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_{25} \\ | & | & & | \end{bmatrix}}_{\text{matrix A,}25\times25} \mathbf{x} = \mathbf{v}$$

This system has been analyzed [AF98]:

- The matrix A is quadratic but *not* invertible.
- Using Gauss-Jordan elimination, we can still solve this system.
- This allows you to win Lights Out whenever this is possible (it isn't always)!

References

Marlow Anderson and Todd Feil.
Turning lights out with linear algebra.
Mathematics Magazine, 71(4):300–303, 1998.
https://doi.org/10.1080/0025570X.1998.11996658.