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Linear Algebra

HS 2025

Assignment 13

Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

There will be no hand-in for this assignment. Solutions will be published on December 22.

Exercises

1. A positive semidefinite matrix (in-class) (★☆☆)

Let $n \in \mathbb{N}^+$ and let $S \in \mathbb{R}^{n\times n}$ such that $S^\top = -S$. Prove that $-S^2$ is symmetric and positive semidefinite.

2. A positive definite matrix (★★☆)

Let $n \in \mathbb{N}^+$ be arbitrary and consider the matrix $A \in \mathbb{R}^{n\times n}$ defined as

$$A := (n-1)I + B$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix and $B \in \mathbb{R}^{n\times n}$ satisfies $B_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, n\}$ (i.e. all entries of B are 1). Prove that A is positive definite.

3. Diagonally dominant matrix (★★☆)

A matrix $A \in \mathbb{R}^{n\times n}$ is called *diagonally dominant* if for every row of the matrix, the absolute value of the diagonal entry in a row is at least the sum of the absolute values of all the other (off-diagonal) entries in that row, i.e.,

$$|A_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |A_{ij}| \qquad \forall i \in \{1, \ldots, n\}$$

Show that a symmetric diagonally dominant matrix $A \in \mathbb{R}^{n\times n}$ with non-negative diagonal entries is positive semidefinite.

4. Positive (semi-)definite matrices (★★☆)

Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Let $\lambda_{\min}^{(A)} \in \mathbb{R}$ be the smallest eigenvalue of A, let $\lambda_{\min}^{(B)} \in \mathbb{R}$ be the smallest eigenvalue of B, and let $\lambda_{\min}^{(A+B)} \in \mathbb{R}$ be the smallest eigenvalue of $A + B$.

a) Prove that $\lambda_{\min}^{(A+B)} \geq \lambda_{\min}^{(A)} + \lambda_{\min}^{(B)}$.

b) Assume that both A and B are positive semidefinite. Prove that $A+B$ is positive semidefinite.

c) Assume that both A and B are positive definite. Prove that $A + B$ is positive definite.

5. Pseudoinverse via SVD (★★☆)

Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank r with singular value decomposition (SVD) $A = U_r \Sigma_r V_r^\top$ with $U_r \in \mathbb{R}^{m \times r}$, $\Sigma_r \in \mathbb{R}^{r \times r}$, and $V_r \in \mathbb{R}^{n \times r}$. Recall that A has a pseudoinverse $A^\dagger$. Note that Σ_r is invertible since it is a square diagonal matrix with non-zero entries on its diagonal. Prove that $A^\dagger = V_r \Sigma_r^{-1} U_r^\top$.

6. Least squares via SVD (★★☆)

In this task, we derive the solution of the least squares method using the singular value decomposition. Let $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$ and $\mathbf{b} \in \mathbb{R}^m$ be arbitrary. Let $A = U \Sigma V^\top$ be an SVD of A with $U \in \mathbb{R}^{m \times m}$, $\Sigma \in \mathbb{R}^{m \times n}$, and $V \in \mathbb{R}^{n \times n}$. Consider the least squares problem

$$\arg\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2. \tag{1}$$

a) Let $\mathbf{c} = U^\top \mathbf{b}$. Prove that $\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2 = \min_{\mathbf{y} \in \mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$.

b) Let $\sigma_1 \geq \cdots \geq \sigma_r$ denote the non-zero singular values of A (r is the rank of A). In particular, we have $\Sigma_{ii} = \sigma_i$ for all $i \in [r]$. Find a formula for the optimal solution $\mathbf{y}^* = \arg\min_{\mathbf{y} \in \mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$ in terms of $\sigma_1, \ldots, \sigma_r$ and $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \ldots & c_m \end{bmatrix}^\top$.

c) Let $\mathbf{x}^*$ be the optimal solution $\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|_2^2$. Given the optimal solution $\mathbf{y}^* = \arg\min_{\mathbf{y} \in \mathbb{R}^n} \|\Sigma\mathbf{y} - \mathbf{c}\|_2^2$ and the SVD of A, how can you compute $\mathbf{x}^*$?