D-INFK
Bernd Gärtner
Robert Weismantel

Linear Algebra

HS 2025

Assignment 3

Submission Deadline: **14 October, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback and bonus points for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Linear functional (in-class) (★☆☆)

a) Let $n \in \mathbb{N}^+$. Consider the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

$$T : \mathbf{x} \mapsto \sum_{k=1}^{n} k x_k$$

for all $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top \in \mathbb{R}^n$. Prove that T is a linear functional.

b) Let $n \in \mathbb{N}^+$ with $n \geq 2$ be arbitrary. Consider the function $T : \mathbb{R}^n \to \mathbb{R}$ defined by

$$T : \mathbf{x} \mapsto \sum_{k=1}^{n} (x_k)^k$$

for all $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top \in \mathbb{R}^n$. Is T a linear functional?

2. Matrix powers (bonus, hand-in) (★☆☆)

For a natural number $k \geq 1$, we define the k-th power of a square matrix A as the matrix multiplication

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Moreover we define $A^0 = I$, where I is the identity matrix.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Use induction to show that for any $k \geq 0$ the k-th power of the matrix A is

$$A^k = \begin{bmatrix} 1+k & k \\ -k & 1-k \end{bmatrix}$$

3. Reconstruct a linear transformation (★☆☆)

a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\\2\end{pmatrix}, T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}2\\3\\2\end{pmatrix}.$$

Determine the general formula for $T\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$ with $x, y \in \mathbb{R}$.

b) Find a matrix A such that $T_A = T$.

4. Linear transformation (★☆☆)

Let $m, n \in \mathbb{N}^+$ and consider an arbitrary $m \times (n+1)$ matrix

$$A = \begin{bmatrix} | & | & & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{v}_{n+1} \\ | & | & & | & | \end{bmatrix}$$

with columns $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{R}^m$. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the function defined by

$$T : \mathbf{x} \mapsto A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

for all $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}^\top \in \mathbb{R}^n$. Prove that T is a linear transformation if and only if $\mathbf{v}_{n+1} = \mathbf{0}$.

5. Matrix multiplication (★★★)

a) Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find $x, y, z \in \mathbb{R}$ such that $A^3 + xA^2 + yA + zI = 0$. Note that both I and 0 are 3×3 matrices in this equation.

b) Let A and B be $m \times m$ matrices. Assume that A and B are commuting, i.e. $AB = BA$. Prove that we have $(AB)^k = A^k B^k$ for all $k \in \mathbb{N}$.

We say that a square matrix A is *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The minimal $k \in \mathbb{N}$ such that $A^k = 0$ is called the *nilpotent degree* of A.

c) Let A be a nilpotent matrix of degree $k \in \mathbb{N}$, and B be a matrix commuting with A. In particular, note that both A and B are square matrices. Is AB nilpotent? If yes, what can we say about the nilpotent degree of AB?

d) Let A be an $m \times m$ nilpotent matrix of degree $k \in \mathbb{N}$. Prove that $(I - A)(I + A + \ldots + A^{k-1}) = I$.

e) Let T be an $m \times m$ upper triangular matrix. Assume that the diagonal of T consists of 0's only. Prove that $T^m = 0$, i.e. T is nilpotent of degree less or equal to m.

Hint: Even if you do not manage to solve a question, you can use its result to tackle subsequent questions.

6. Rotation matrices (★★☆)

Hint: This exercise requires some basic knowledge of sin *and* cos*. Part c) can also be solved independently by assuming parts a) and b).*

a) We define a real 2×2 matrix A to be a *rotation matrix* if and only if there exists a rotation angle $\phi \in \mathbb{R}$ such that

$$A = Q(\phi) \coloneqq \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Prove that the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a rotation matrix according to this definition.

b) Show that the matrix product $Q(\phi_1)\,Q(\phi_2)$ of two rotation matrices with angles ϕ_1 and ϕ_2 is again a rotation matrix $Q(\phi_3)$ according to this definition, and determine the corresponding rotation angle ϕ_3.

Hint: You might need to review trigonometric formulas to solve this question.

c) Let A be a 2×2 rotation matrix. Prove that there exists a 2×2 matrix B such that $AB = BA = I$.