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Assignment 4

Submission Deadline: **21 October, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 3 by handing in your solution as pdf via Moodle before the deadline.

1. Matrix multiplication with vectors and covectors (in-class) (★☆☆)

Let $n \in \mathbb{N}$. Consider $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ given by $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \end{pmatrix}$.

a) Compute $\mathbf{v}^\top \mathbf{w}$ for all $n \in \mathbb{N}$.

b) Compute $\mathbf{v}\mathbf{w}^\top$ when $n = 4$.

c) Compute $\mathbf{w}^\top(\mathbf{v}\mathbf{w}^\top)\mathbf{v}$ for all $n \in \mathbb{N}$.

2. Exercise 2.47 (in-class) (★☆☆)

Let $A \in \mathbb{R}^{m\times n}$ of rank r and $C \in \mathbb{R}^{m\times r}$ and $R' \in \mathbb{R}^{r\times n}$ be the matrices in the CR-decomposition of A as given in Theorem 2.46.

a) Suppose $r = n$. What are the matrices C and R'?

b) Suppose $r = 0$. What are the matrices C and R'?

3. Matrix multiplication and invertibility (hand-in) (★★☆)

Let $A, B, C \in \mathbb{R}^{m\times m}$ such that $BA = CA$.

a) Suppose that A is invertible. Show that $B = C$.

b) Is it true that $AB = AC$? Give either a proof or a counterexample.

c) Suppose that $B - C$ is invertible. Show that $A = 0$.

4. Special matrix inverses (★★☆)

a) What is the inverse A^{-1} of the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$?

b) What is the inverse D^{-1} of the diagonal matrix $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$?

c) What is the inverse B^{-1} of the matrix $B = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{bmatrix}$?

Hint: Try to undo the matrix transformations T_A, T_B, T_D, i.e., find their inverses and the corresponding matrices.

5. Inverses of matrix powers (★★☆)

a) Let A be an $m \times m$ matrix with inverse A^{-1} and let $k \in \mathbb{N}^+$ be an arbitrary integer. Does A^k have an inverse and if yes, what is it?

b) Recall the definition of a nilpotent matrix: We say that a square matrix A is nilpotent if and only if there exists $k \in \mathbb{N}$ such that $A^k = 0$. Prove that a nilpotent matrix A cannot have an inverse.

c) Let A be an $m \times m$ matrix with $A^3 = I$ and $A^4 = I$. Prove that $A = I$.

d) Find a 2×2 matrix $A \neq I$ such that $A^k = I$ for all even k and $A^k = A$ for all odd $k \in \mathbb{N}$.

e) Can you also find a 2×2 matrix A that, for all $k \in \mathbb{N}$, satisfies $A^k = I$ if and only if $k \equiv_4 0$ (i.e. $A^k = I$ if and only if k is a multiple of 4)?

6. Inverse of triangular matrices (★★★)

a) Find the inverse of the 2×2 matrix $L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ where $a \in \mathbb{R}$.

b) Prove that a square lower triangular matrix is invertible if and only if all its diagonal entries are non-zero.

c) Prove that the inverse of any lower triangular matrix, if it exists, is lower triangular itself.

d) Are the statements of b) and c) also true if we replace *lower triangular* by *upper triangular*?