D-INFK
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Linear Algebra

HS 2025

Assignment 5

Submission Deadline: **28 October, 2025** at 23:59

Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. **Interpolation (in-class)** (★☆☆) Assume that you have collected the following data points

x	y
0	1
2	2
4	5
6	6

and you want to find a function $f : \mathbb{R} \to \mathbb{R}$ that interpolates them, i.e. f should satisfy $f(x) = y$ for all pairs of x, y given by the table above. There is an abundance of functions that you can try and in particular, there are many different functions that do interpolate the four datapoints. In this exercise, we are interested in polynomials, i.e. we restrict f to be a polynomial of degree at most 3. In particular, this means that f has the form $f(x) = ax^3 + bx^2 + cx + d$ for some $a, b, c, d \in \mathbb{R}$. Your task is to find values for a, b, c, d such that f interpolates all four points given in the table.

2. **Strictly diagonally dominant matrices (hand-in)** (★★★)

Let $A \in \mathbb{R}^{m\times m}$ be strictly diagonally dominant, that is,

$$|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}|$$

holds for all $i \in [m]$. Prove that A is invertible.

3. **Solving linear systems** (★☆☆) Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most 2, i.e. $p(x) = ax^2 + bx + c$ for some coefficients $a, b, c \in \mathbb{R}$. Assume that we already know $p(-1) = 0$, $p(0) = 2$ and $p(1) = 2$. Find the coefficients a, b and c. As the title suggests, you will have to solve a linear system. We recommend that you do it by using the systematic elimination procedure from the lecture.

4. **Invertibility** (★☆☆)

Let $A \in \mathbb{R}^{3\times 3}$ be the following upper triangular matrix with $a, b, c, d \in \mathbb{R}$:

$$A = \begin{pmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix}.$$

For which values of a, b, c, d is A invertible? Specify A^{-1} for these cases.

5. Inverse (★★☆)

a) Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

Justify your answer.

b) Find the inverse of the matrix $A \in \mathbb{R}^{m\times m}$ defined by

$$a_{ij} := \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

for all $i, j \in [m]$. Justify your answer. Note that this is a generalization of subtask a), where you were asked to solve the special case $m = 4$.

6. Linear independence in $\mathbb{R}^3$ (★★☆)

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be three linearly independent vectors in $\mathbb{R}^3$. Consider the three vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \in \mathbb{R}^3$ defined as

$$\mathbf{w}_1 := \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_2 := -\mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 := \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3.$$

Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent.