Assignment 6

Submission Deadline: **04 November, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA and bonus points for Exercise 3 by handing in your solution as pdf via Moodle before the deadline.

1. Gauss-Jordan and CR decomposition (★☆☆)

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & -1 \\ -1 & 3 & 2 & 4 \end{bmatrix}.$$

Compute the CR decomposition of A using Gauss-Jordan elimination.

2. Subspaces of function spaces and $\mathbb{R}^{m \times m}$ (in-class) (★☆☆)

a) In this exercise we consider the vector space V of all real-valued functions on the interval $[0,1]$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : [0,1] \to \mathbb{R}$ and conversely, every function $\mathbf{f} : [0,1] \to \mathbb{R}$ is in V. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

$$(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in [0,1]$$
$$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in [0,1] \text{ and } c \in \mathbb{R}.$$

Prove that

$$U = \{\mathbf{f} \in V : \mathbf{f}(x) = \mathbf{f}(1-x) \text{ for all } x \in [0,1]\} \subseteq V$$

is a subspace of V.

b) Let $m \in \mathbb{N}^+$. Consider the set $\mathcal{D}_m$ of diagonal $m \times m$ matrices, which is a subspace of $\mathbb{R}^{m \times m}$. What is the dimension of $\mathcal{D}_m$? Justify your answer with a proof.

3. Skew-symmetric matrices as a subspace (bonus, hand-in) (★★☆)

Let $m \in \mathbb{N}^+$. A matrix $A \in \mathbb{R}^{m \times m}$ is skew-symmetric if and only if $A = -A^\top$. Consider the set $\mathcal{S}_m$ of skew-symmetric $m \times m$ matrices.

a) Show that $\mathcal{S}_m$ is a subspace of $\mathbb{R}^{m \times m}$.

b) What is the dimension of $\mathcal{S}_m$? Justify your answer with a proof.

Hint: You can use Assignment 2 Exercise 6b) without a proof.

4. Subspace of univariate polynomials (★☆☆)

Consider the three polynomials $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbb{R}[x]$ defined as

$$\mathbf{p} = x^3 + x, \quad \mathbf{q} = x^2 + 1, \quad \mathbf{r} = x^2 + x + 1.$$

What is the dimension of $\mathsf{Span}(\mathbf{p}, \mathbf{q}, \mathbf{r}) \subseteq \mathbb{R}[x]$? Prove your answer.

5. Subspaces of $\mathbb{R}^m$ and $\mathbb{R}^{2\times m}$ (★★☆)

a) Let H be a hyperplane through the origin of $\mathbb{R}^m$. Recall that this means that there exists a non-zero vector $\mathbf{d} \in \mathbb{R}^m$ with $H = \{\mathbf{v} \cdot \mathbf{d} = 0 : \mathbf{v} \in \mathbb{R}^m\}$. Prove that H is a subspace of $\mathbb{R}^m$.

b) Consider again a hyperplane H through the origin of $\mathbb{R}^m$. Prove that the dimension of H is $m - 1$.

c) Let $m \in \mathbb{N}^+$. Fix an arbitrary non-zero vector $\mathbf{v} \in \mathbb{R}^m$ and consider the set of matrices $S^{\mathbf{v}} := \{A \in \mathbb{R}^{2\times m} : A\mathbf{v} = \mathbf{0}\} \subseteq \mathbb{R}^{2\times m}$. It is not hard to show that $S^{\mathbf{v}}$ is a subspace of $\mathbb{R}^{2\times m}$ (you do not have to show this, you can assume it without proof). What is the dimension of $S^{\mathbf{v}}$? Prove your answer.

Hint: *Use the statements from parts a) and b).*

6. Union of subspaces (★★☆)

Let V be a vector space and let U and W be subspaces of V. Show that $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

7. Odd and even functions (★★★)

In this exercise, we consider the vector space V of all real-valued functions on $\mathbb{R}$. In other words, every element $\mathbf{f} \in V$ is a function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ and conversely, every function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is in V. Note that it might not be obvious that this is a vector space, but for the purpose of this exercise you can assume that it is. In particular, there exists a valid addition $\mathbf{f} + \mathbf{g}$ of such functions $\mathbf{f} \in V$ and $\mathbf{g} \in V$, and a valid scalar multiplication $c\mathbf{f}$ for a scalar $c \in \mathbb{R}$ and $\mathbf{f} \in V$ defined as follows:

$$(\mathbf{f} + \mathbf{g})(x) := \mathbf{f}(x) + \mathbf{g}(x) \qquad \text{for all } \mathbf{f}, \mathbf{g} \in V \text{ and } x \in \mathbb{R}$$
$$(c\mathbf{f})(x) := c\mathbf{f}(x) \qquad \text{for all } \mathbf{f} \in V, x \in \mathbb{R} \text{ and } c \in \mathbb{R}.$$

Now consider the set of odd functions

$$O = \{\mathbf{f} \in V : \mathbf{f}(-x) = -\mathbf{f}(x) \text{ for all } x \in \mathbb{R}\}$$

and the set of even functions

$$E = \{\mathbf{f} \in V : \mathbf{f}(-x) = \mathbf{f}(x) \text{ for all } x \in \mathbb{R}\}\,.$$

a) Prove that both O and E are subspaces of V.

b) Prove that the intersection $O \cap E$ contains only the zero function $\mathbf{0} : \mathbb{R} \to \mathbb{R}$ with $\mathbf{0}(x) = 0$ for all $x \in \mathbb{R}$.

c) Prove that any function $\mathbf{f} \in V$ can be written as $\mathbf{f} = \mathbf{g} + \mathbf{h}$ for some $\mathbf{g} \in E$ and $\mathbf{h} \in O$.