D-INFK Linear Algebra HS 2025
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Assignment 7

Submission Deadline: **11 November, 2025** at 23:59
Course Website: https://ti.inf.ethz.ch/ew/courses/LA25/index.html

Exercises

You can get feedback from your TA for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. **Solving linear systems (in-class)** (★☆☆)

   Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} -1 & 2 & 5 & -2 \\ -3 & 3 & 12 & -3 \\ 1 & -14 & -7 & -6 \end{bmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} -6 \\ -15 \\ 8 \end{pmatrix}.$$

   a) Determine the set of solutions $\mathbf{Sol}(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{b}\}$, i.e., write down an *explicit* characterization of this set of solutions in the form presented after Theorem 4.38 on page 155 in the lecture notes.

   b) Write down a basis for $\mathbf{N}(A)$ (you might have already found it in the previous subtask), and also find a basis for $\mathbf{C}(A)$.

   c) What are the dimensions of $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$, and $\mathbf{R}(A)$?

   d) Determine a basis of $\mathbf{R}(A)$.

2. **Nullspace and column space (hand-in)** (★★☆)

   Let $\mathbf{v}$ be a *unit vector* (i.e. $\| \mathbf{v} \| = 1$) in $\mathbb{R}^3$. Consider the 3×3 matrices A and P defined by

$$A := \mathbf{v}\mathbf{v}^\top, \qquad P := I_3 - \mathbf{v}\mathbf{v}^\top = I_3 - A$$

   where I_3 is the 3×3 identity matrix.

   a) Show that $A^2 = A$ and $P^2 = P$.

   b) Let $\mathbf{w} \in \mathbb{R}^3$ be orthogonal to $\mathbf{v}$. Prove $A\mathbf{w} = \mathbf{0}$.

   c) Now let $\mathbf{w} \in \mathbb{R}^3$ be a vector satisfying $A\mathbf{w} = \mathbf{0}$. Prove $\mathbf{w} \cdot \mathbf{v} = 0$.

   d) Based on b) and c), describe the nullspace $\mathbf{N}(A)$.

   e) Determine the rank of A. Is A invertible?

   f) Prove that $\mathbf{C}(A) = \{\alpha\mathbf{v} : \alpha \in \mathbb{R}\} = \mathsf{Span}(\mathbf{v})$.

   g) Also prove that $\mathbf{C}(A) = \{\mathbf{w} \in \mathbb{R}^3 : A\mathbf{w} = \mathbf{w}\}$.

   h) Use g) to prove $\mathbf{N}(P) = \mathbf{C}(A)$.

i) Finally, prove $\mathbf{C}(P) = \mathbf{N}(A)$.

Hint: In every subtask you may of course use statements that you have already proven in previous subtasks. For some of the subtasks we specifically tell you which previous subtasks might be helpful.

3. Reconstruct a matrix (★☆☆)

Let A be a 3×2 matrix satisfying

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ and } A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

a) Determine A.

b) Determine the dimensions of the three fundamental subspaces $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{R}(A)$ of A.

4. Orthogonality and triangle inequality (★☆☆)

a) Show that two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal if and only if

$$\| \mathbf{v} + \mathbf{w} \|^2 = \| \mathbf{v} \|^2 + \| \mathbf{w} \|^2 .$$

b) Prove or provide a counterexample: Three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are pairwise orthogonal if and only if

$$\| \mathbf{u} + \mathbf{v} + \mathbf{w} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2 + \| \mathbf{w} \|^2 .$$

5. Orthogonal subspaces in $\mathbb{R}^4$ (★★★)

Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$ and $W = \left\{ \mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0 \right\} \subseteq \mathbb{R}^4$ be subspaces of $\mathbb{R}^4$.

a) Determine a basis of $V^\perp$.

b) Does there exist a basis of W such that no basis element is in $V^\perp$?

c) Does there exist a basis of W such that exactly one basis element is in $V^\perp$?

d) Does there exist a basis of W such that exactly two basis elements are in $V^\perp$?

e) Does there exist a basis of W such that exactly three basis elements are in $V^\perp$?

Justify your answers.

6. **1.** Which of the following statements is true for all $m \times m$ matrices A?

(**a**) $\mathbf{N}(A) = \mathbf{N}(2A)$

(**b**) $\mathbf{N}(A) = \mathbf{N}(A^2)$

(**c**) $\mathbf{N}(A) = \mathbf{N}(A + I)$

(**d**) $\mathbf{N}(A) = \mathbf{N}(A^\top)$

2. Which of the following statements is true for all square matrices A?

(**a**) $\mathbf{C}(A) = \mathbf{C}(2A)$

(**b**) $\mathbf{C}(A) = \mathbf{C}(A^2)$

(**c**) $\mathbf{C}(A) = \mathbf{C}(A + I)$

(**d**) $\mathbf{C}(A) = \mathbf{C}(A^\top)$

3. The following equations each describe a plane in $\mathbb{R}^3$:

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 0 \\ 2x & - & 5y & + & 3z & = & 0 \\ 3x & & & + & 4z & = & 0. \end{array}$$

Which of the following statements is true?

(**a**) The intersection of all three planes is empty.

(**b**) The intersection of all three planes contains exactly one element.

(**c**) The intersection of all three planes is a line.

4. Consider the linear system

$$x_1 + (b-1)x_2 = 3$$
$$-3x_1 - (2b-8)x_2 = -5$$

with variables x_1, x_2 and parameter $b \in \mathbb{R}$. For which values of b is the set of solutions to the above system empty (i.e. there is no solution)?

(**a**) Only for $b = 0$.

(**b**) Only for $b = -5$.

(**c**) For all possible values of b (i.e. for all of $\mathbb{R}$).

(**d**) The system always has a solution regardless of the value of b.