Solution for Assignment 11

1. **a)** Let C_{ij} be the co-factors of A where $i, j \in [5]$. Note that by combining Theorem 7.2.5 and Proposition 7.3.2, we get

$$\begin{aligned}
\mathsf{det} A &\overset{7.2.5}{=} \mathsf{det} A^\top \\
&\overset{7.3.2}{=} \sum_{j=1}^{5} (A^\top)_{3,j} (C^\top)_{3,j} \\
&= \sum_{i=1}^{5} A_{i,3} C_{i,3} \\
&= 0C_{1,3} + 0C_{2,3} + bC_{3,3} + 0C_{4,3} + 0C_{5,3} \\
&= b \cdot (-1)^{(3+3)} \cdot \begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix}.
\end{aligned}$$

This is also sometimes called *expansion of the determinant along the third column*. In particular, we chose the third column because it contains many zeroes and hence many terms disappeared. In order to compute

$$\begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix}$$

we use the same trick again for the first column. In this way we obtain

$$\begin{vmatrix} 0 & 1 & 4 & c \\ a & 5 & 4 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & -4 & 3 & 1 \end{vmatrix} = a \cdot (-1)^{(2+1)} \begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix}.$$

We repeat this one more time for the third column of

$$\begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix}$$

to get

$$\begin{vmatrix} 1 & 4 & c \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{vmatrix} = c \cdot (-1)^{(1+3)} \cdot \begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} + 1 \cdot (-1)^{(3+3)} \cdot \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix}.$$

We can compute these 2×2 determinants directly with the formula from the lecture as

$$\begin{vmatrix} -2 & 1 \\ -4 & 3 \end{vmatrix} = -2 \text{ and } \begin{vmatrix} 1 & 4 \\ -2 & 1 \end{vmatrix} = 9.$$

Putting everything together, we obtain

$$\mathsf{det} A = b \cdot (-1)^{(3+3)} \left(a \cdot (-1)^{(2+1)} \left(c \cdot (-1)^{(1+3)} \cdot (-2) + 1 \cdot (-1)^{(3+3)} \cdot 9 \right) \right) = ab(2c - 9).$$

We conclude that $\mathsf{det} A = 0$ if and only if $a = 0$, or $b = 0$, or $c = \frac{9}{2}$.

b) As it turns out, we only need to perform one step of Gauss elimination on B to obtain U:

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & 0 \\ -1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & 6 \\ 0 & 0 & -1 \end{bmatrix} =: U.$$

Using Proposition 7.2.4, we see that $\mathsf{det}(U) = -2$. Following the discussion in Section 7.3 after Proposition 7.3.5, we know that the determinant of U is the same as the determinant of B (we did not swap any rows). Hence, we conclude $\mathsf{det}(B) = -2$.

2. **a)** This is a solution for a) that uses the decomposition in the hint. Further below, we provide an alternative solution that does not use the hint. Observe that, as suggested in the hint, we can decompose M as

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & C \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

where we used identity matrices I and zero matrices 0 of the appropriate dimensions. By Theorem 7.2.6, we have

$$\mathsf{det}(M) = \mathsf{det} \begin{bmatrix} I & B \\ 0 & C \end{bmatrix} \mathsf{det} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}.$$

Consider first the matrix $M' := \begin{bmatrix} I & B \\ 0 & C \end{bmatrix}$. Observe that there is a unique non-zero entry in each of its first m columns. Thus, every permutation σ that contributes to the determinant of M' in the formula

$$\mathsf{det} M' = \sum_{\sigma \in \Pi_{\mathrm{n}}} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M'_{i,\sigma(i)}$$

must select these non-zero entries, i.e. $\sigma(i) = i$ for all $i \in [m]$. The formula then simplyfies to

$$\mathsf{det} M' = \sum_{\sigma \in \Pi_{\mathrm{n-m}}} \mathsf{sign}(\sigma) \prod_{i=1}^{n-m} C_{i,\sigma(i)} = \mathsf{det} C$$

as the non-zero entries in the first m columns (or rows) are all one.

Analogously, we get

$$\mathsf{det} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \sum_{\sigma \in \Pi_{\mathrm{m}}} \mathsf{sign}(\sigma) \prod_{i=1}^{m} A_{i,\sigma(i)} = \mathsf{det} A$$

and thus we conclude $\mathsf{det} M = \mathsf{det}(A)\mathsf{det}(C)$.

a') Here is a solution that ignores the decomposition in the hint: We start by using the definition of the determinant for M, i.e. we have

$$\mathsf{det} M = \sum_{\sigma \in \Pi_{\mathrm{n}}} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$$

where Π_{n} is the set of all permutations on n elements. The key observation for this exercise is that only those permutations $\sigma \in \Pi_{\mathrm{n}}$ that satisfy $\sigma(1), \dots, \sigma(m) \in \{1, \dots, m\}$ will contribute to this sum. To see this, let $\sigma \in \Pi_{\mathrm{n}}$ be a permutation with $\sigma(i) > m$ for some $i \in [m]$. By the pigeonhole principle, there must exist $j \in [n] \setminus [m]$ with $\sigma(j) \in [m]$. But by the shape of M, we must have $M_{j,\sigma(j)} = 0$ and hence the contribution of σ to the sum is 0.

In particular, the relevant (those that contribute non-zero terms to the sum) permutations $\sigma \in \Pi_{\mathrm{n}}$ satisfy $\sigma(i) \in [m]$ for all $i \in [m]$ and $\sigma(j) \in [n] \setminus [m]$ for all $j \in [n] \setminus [m]$. In other words, restricting such a permutation σ to $[m]$ yields a permutation on m elements, and restricting σ to $[n] \setminus [m]$ yields a permutation on $n - m$ elements. Conversely, any two permutations $\sigma_1 \in \Pi_\mathrm{m}$ and $\sigma_2 \in \Pi_\mathrm{n-m}$ yield a permutation $\sigma \in \Pi_\mathrm{n}$ that contributes to the sum (define $\sigma(i) = \sigma_1(i)$ for $i \in [m]$ and $\sigma(j) = m + \sigma_2(j - m)$ for $j \in [n] \setminus [m]$). Observe that the number of inversions in σ is exactly the number of inversions in σ_1 plus the number of inversions in σ_2. Hence, we always have $\mathsf{sign}(\sigma) = \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2)$ in this correspondence.

We conclude that we can rewrite the sum as

$$\mathsf{det}M = \sum_{\sigma \in \Pi_\mathrm{n}} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)} = \sum_{\sigma_1 \in \Pi_\mathrm{m}} \sum_{\sigma_2 \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2) \prod_{i=1}^{m} M_{i,\sigma_1(i)} \prod_{j=m+1}^{n} M_{j,\, j+\sigma_2(j-m)}.$$

Next, observe that the terms $M_{i,\sigma_1(i)}$ are always in the A-part of M, i.e. we have $M_{i,\sigma_1(i)} = A_{i,\sigma_1(i)}$. Similarly, the terms $M_{j,\, j+\sigma_2(j-m)}$ are always in the C-part of M, i.e. we have $M_{j,\, j+\sigma_2(j-m)} = C_{j-m,\sigma_2(j-m)}$. Hence, we can further rewrite the sum as

$$\begin{aligned}
\mathsf{det}M &= \sum_{\sigma_1 \in \Pi_\mathrm{m}} \sum_{\sigma_2 \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2) \prod_{i=1}^{m} M_{i,\sigma_1(i)} \prod_{j=m+1}^{n} M_{j,\, j+\sigma_2(j-m)} \\
&= \sum_{\sigma_1 \in \Pi_\mathrm{m}} \sum_{\sigma_2 \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma_1)\mathsf{sign}(\sigma_2) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \prod_{j=m+1}^{n} C_{j-m,\, \sigma_2(j-m)} \\
&= \sum_{\sigma_1 \in \Pi_\mathrm{m}} \mathsf{sign}(\sigma_1) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \left(\sum_{\sigma_2 \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma_2) \prod_{j=m+1}^{n} C_{j-m,\, \sigma_2(j-m)} \right) \\
&= \left(\sum_{\sigma_1 \in \Pi_\mathrm{m}} \mathsf{sign}(\sigma_1) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \right) \left(\sum_{\sigma_2 \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma_2) \prod_{j=m+1}^{n} C_{j-m,\, \sigma_2(j-m)} \right) \\
&= \left(\sum_{\sigma_1 \in \Pi_\mathrm{m}} \mathsf{sign}(\sigma_1) \prod_{i=1}^{m} A_{i,\sigma_1(i)} \right) \left(\sum_{\sigma_2 \in \Pi_\mathrm{n-m}} \mathsf{sign}(\sigma_2) \prod_{j=1}^{n-m} C_{j,\, \sigma_2(j)} \right) \\
&= \mathsf{det}(A)\mathsf{det}(C)
\end{aligned}$$

which concludes the proof.

b) In order to calculate the determinant of M using the previous result, we must first bring it into the right form. Clearly, M already contains a lot of zero entries. In the end, we want to have a block of zeroes in the bottom left corner. We can use that transposing the matrix does not change its determinant. Moreover, by Proposition 7.3.6, swapping two rows of a matrix negates its determinant. Hence we proceed as follows: we first transpose M and then swap the second row and fourth row, as well as the third and sixth row of the resulting matrix. In this way, we obtain the matrix

$$M' = \begin{bmatrix} 2 & 9 & 1 & 3 & 2 & 8 \\ 4 & 0 & 0 & 5 & 5 & 3 \\ 7 & 4 & 0 & 7 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Using the result from the previous subtask and some more row swaps as well as the formula for the determinant of triangular matrices, we get

$$\begin{aligned}
\mathsf{det}M &= (-1)^2 \mathsf{det}M' \\
&= \mathsf{det}M' \\
&= \begin{vmatrix} 2 & 9 & 1 \\ 4 & 0 & 0 \\ 7 & 4 & 0 \end{vmatrix} \begin{vmatrix} 2 & 3 & 8 \\ 0 & 0 & 2 \\ 0 & 1 & 7 \end{vmatrix} \\
&= (-1)^2 \begin{vmatrix} 4 & 0 & 0 \\ 7 & 4 & 0 \\ 2 & 9 & 1 \end{vmatrix} (-1) \begin{vmatrix} 2 & 3 & 8 \\ 0 & 1 & 7 \\ 0 & 0 & 2 \end{vmatrix} \\
&= -16 \cdot 4 = -64.
\end{aligned}$$

3. Using the rules we learned in the lecture, we calculate

$$\begin{aligned}
u + v + w &= (u + v) + w = (4 + 2i) + (3 - 4i) = (4 + 3) + (2 - 4)i = 7 - 2i \\
u \cdot v &= (3 + i) \cdot (1 + i) = 3 + 3i + i - 1 = 2 + 4i \\
v \cdot w \cdot i &= (1 + i) \cdot (3 - 4i) \cdot i = (3 - 4i + 3i + 4) \cdot i = 3i + 4 - 3 + 4i = 1 + 7i \\
\frac{w}{v} &= \frac{w}{v} \cdot \frac{\overline{v}}{\overline{v}} = \frac{(3 - 4i)(1 - i)}{(1 + i)(1 - i)} = \frac{3 - 3i - 4i - 4}{1 + 1} = -\frac{1}{2} - \frac{7}{2}i \\
\frac{v}{u} &= \frac{v}{u} \cdot \frac{\overline{u}}{\overline{u}} = \frac{(1 + i)(3 - i)}{(3 + i)(3 - i)} = \frac{3 - i + 3i + 1}{9 + 1} = \frac{2}{5} + \frac{1}{5}i \\
|v| &= \sqrt{1^2 + 1^2} = \sqrt{2}.
\end{aligned}$$

4. In this exercise we want to exploit Proposition 7.3.7 which says that the determinant is linear in each row. In particular, using this on the second row of A and B, we get

$$\mathsf{det}(A) - \mathsf{det}(B) = \mathsf{det}\begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & \mathbf{u}_1^\top & - \\ & M & \end{bmatrix} - \mathsf{det}\begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & \mathbf{u}_2^\top & - \\ & M & \end{bmatrix} = \mathsf{det}\begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & (\mathbf{u}_1 - \mathbf{u}_2)^\top & - \\ & M & \end{bmatrix}.$$

Analogously, we can use it on the second row of C and D to get

$$\mathsf{det}(C) - \mathsf{det}(D) = \mathsf{det}\begin{bmatrix} - & \mathbf{v}_2^\top & - \\ - & \mathbf{u}_1^\top & - \\ & M & \end{bmatrix} - \mathsf{det}\begin{bmatrix} - & \mathbf{v}_2^\top & - \\ - & \mathbf{u}_2^\top & - \\ & M & \end{bmatrix} = \mathsf{det}\begin{bmatrix} - & \mathbf{v}_2^\top & - \\ - & (\mathbf{u}_1 - \mathbf{u}_2)^\top & - \\ & M & \end{bmatrix}.$$

Finally, using linearity in the first row of those two resulting matrices yields

$$\mathsf{det}\begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & (\mathbf{u}_1 - \mathbf{u}_2)^\top & - \\ & M & \end{bmatrix} - \mathsf{det}\begin{bmatrix} - & \mathbf{v}_2^\top & - \\ - & (\mathbf{u}_1 - \mathbf{u}_2)^\top & - \\ & M & \end{bmatrix} = \mathsf{det}\begin{bmatrix} - & (\mathbf{v}_1 - \mathbf{v}_2)^\top & - \\ - & (\mathbf{u}_1 - \mathbf{u}_2)^\top & - \\ & M & \end{bmatrix}$$

and thus

$$\mathsf{det}(A) - \mathsf{det}(B) - \mathsf{det}(C) + \mathsf{det}(D) = \mathsf{det}(E).$$

5. a) Let $\lambda \in \mathbb{R}$ be an arbitrary real eigenvalue of M with corresponding real eigenvector $\mathbf{v} \in \mathbb{R}^n$, i.e. we have

$$M\mathbf{v} = \lambda\mathbf{v}.$$

Now let's see what happens to $\mathbf{v}$ if we apply $M + cI$ instead of M to it:

$$\begin{aligned}
(M + cI)\mathbf{v} &= M\mathbf{v} + c\mathbf{v} \\
&= \lambda\mathbf{v} + c\mathbf{v} \\
&= (\lambda + c)\mathbf{v}.
\end{aligned}$$

As we have observed, $\mathbf{v}$ is a real eigenvector of $M + cI$ with corresponding real eigenvalue $c + \lambda$. This is exactly what we wanted to prove.

b) Consider the matrix

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & 3 & 5 & 7 & 9 & 11 \end{bmatrix}.$$

We observe that $A = B + 2I$. Hence, our plan is to find two distinct real eigenvalues of B, and then use the result from the previous subtask.

Since all rows of B are equal, the matrix has rank 1. Thus, 0 is an eigenvalue of B. It remains to find another real eigenvalue. For this, let us try to guess a real eigenvector of B that does not correspond to eigenvalue 0. This is not as hard as it may sound: every row of B is the same, hence any eigenvector of B that does not correspond to eigenvalue 0 should have the same value in each coordinate. Indeed, we have

$$B \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 36 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the vector $\mathbf{1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$ is an eigenvector of B with corresponding eigenvalue 36.

By the result from the previous subtask, it follows that $\lambda_1 = 2$ and $\lambda_2 = 38$ are two distinct real eigenvalues of A.

6. We have to show that PQ is the permutation matrix associated to $q \circ p$. Let $T = PQ$, then we have to show that

$$T_{ik} = \begin{cases} 1 & \text{if } q\big(p(i)\big) = k \\ 0 & \text{otherwise} \end{cases}$$

Let $i \in \{1, \ldots, n\}$ and $\ell := p(i)$. Then we have

$$P_{ij} = \begin{cases} 1 & \text{if } j = p(i) = \ell \\ 0 & \text{otherwise} \end{cases}$$

Thus, for any $k \in \{1, \ldots, n\}$ we have

$$T_{ik} = \sum_{j=1}^{n} P_{ij} Q_{jk} = P_{i\ell} Q_{\ell k} = Q_{\ell k}$$

Therefore

$$T_{ik} = Q_{\ell k} = \begin{cases} 1 & \text{if } k = q(\ell) = q(p(i)) \\ 0 & \text{otherwise} \end{cases}$$

In conclusion, we have shown that the matrix PQ is the permutation matrix associated to the permutation $q \circ p$.