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Solution for Assignment 5

1. Each of the four points yields one linear equation with variables a, b, c, d. For example, for $x = 4, y = 5$ we get the equation

$$a4^3 + b4^2 + c4 + d = 5.$$

In total, we get the linear system

$$\begin{aligned} a0^3 + b0^2 + c0 + d &= 1 \\ a2^3 + b2^2 + c2 + d &= 2 \\ a4^3 + b4^2 + c4 + d &= 5 \\ a6^3 + b6^2 + c6 + d &= 6 \end{aligned}$$

with four equations and four variables that we can write in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 6 & 36 & 216 \end{bmatrix} \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 6 \end{pmatrix}.$$

We now want to solve this system by using the elimination technique. For this, it is convenient to apply the row operations to the system matrix and the right-hand side simultaneously by appending the right-hand side to the matrix as follows:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 4 & 8 & 2 \\ 1 & 4 & 16 & 64 & 5 \\ 1 & 6 & 36 & 216 & 6 \end{array}\right].$$

After performing elimination in the first column we get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 4 & 16 & 64 & 4 \\ 0 & 6 & 36 & 216 & 5 \end{array}\right].$$

Next, we perform elimination in the second columns to get

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 24 & 192 & 2 \end{array}\right].$$

Finally, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 4 & 8 & 1 \\ 0 & 0 & 8 & 48 & 2 \\ 0 & 0 & 0 & 48 & -4 \end{array}\right].$$

It remains to perform backward substitution. From the last row, we get $a = -\frac{4}{48} = -\frac{1}{12}$. Next, we get $b = \frac{2-48a}{8} = \frac{6}{8} = \frac{3}{4}$. From the second row we obtain $c = \frac{1-8a-4b}{2} = \frac{1+\frac{2}{3}-3}{2} = -\frac{2}{3}$. Finally, we get $d = 1$ from the first row. Hence, the function $f(x) = -\frac{1}{12}x^3 + \frac{3}{4}x^2 - \frac{2}{3}x + 1$ interpolates all of our datapoints.

2. We will show that $A\mathbf{x} \neq \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} \neq \mathbf{0}$. Once this is established the claim follows since Corollary 1.23(iii) tells us that the columns of A are linearly independent, which means that A is invertible by Definition 2.55.

Let $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} \neq \mathbf{0}$ and x_i a coordinate of $\mathbf{x}$ that satisfies $|x_i| = \max\{|x_1|, \dots, |x_m|\}$. Consider the i-th entry of $A\mathbf{x}$, which is

$$\sum_{j=1}^{m} a_{ij}x_j = \underbrace{\sum_{\substack{j=1\\ j\neq i}} a_{ij}x_j}_{=:y} + \underbrace{a_{ii}x_i}_{=:z}.$$

We will show that $y + z \neq 0$. To do so, we calculate

$$\begin{aligned} y = \sum_{\substack{j=1\\ j\neq i}} a_{ij}x_j &\leq \sum_{\substack{j=1\\ j\neq i}} |a_{ij}x_j| = \sum_{\substack{j=1\\ j\neq i}} |a_{ij}||x_j| \\ &\leq \sum_{\substack{j=1\\ j\neq i}} |a_{ij}||x_i| \\ &< |a_{ii}||x_i| = |a_{ii}x_i| = |z|. \end{aligned}$$

The first line break follows from the choice of x_i. The second line break is the definition of strictly diagonally dominant matrices. We established $y < |z|$. With a similar calculation, one can show that $-y < |z|$. In total, we have $|y| < |z|$. However, this implies that $0 \neq y + z = \sum_{j=1}^{m} a_{ij}x_j = \sum_{\substack{j=1\\ j\neq i}} a_{ij}x_j$ and the claim follows.

3. Consider the three constraints $p(-1) = 0$, $p(0) = 2$ and $p(1) = 2$ that we have on p. Each of these constraints gives us an equation involving the unknowns a, b and c. In particular, we get the three equations

$$\begin{aligned} a - b + c &= 0 \qquad && \text{from } p(-1) = 0 \\ c &= 2 && \text{from } p(0) = 2 \\ a + b + c &= 2 && \text{from } p(1) = 2 \end{aligned}$$

that we can also write down in matrix form

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

In order to solve this system, let us use the elimination method from the lecture. For this, let us define

$$A = \begin{bmatrix} \mathbf{1} & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

We already highlighted the first pivot in A. After one step of elimination, we get

$$E_{21}A = \begin{bmatrix} \mathbf{1} & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } E_{21}b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

with $E_{21} = I$ as we already had $a_{21} = 0$. In the second step, we obtain

$$E_{31}E_{21}A = \begin{bmatrix} \mathbf{1} & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \text{ and } E_{31}E_{21}b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

with $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Next, we need to permute rows 2 and 3 with the matrix $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ in order to get our next pivot. We obtain

$$P_{23}E_{31}E_{21}A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } P_{23}E_{31}E_{21}b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

In the last elimination step we again find that we do not need to do anything. In other words, we have $E_{32} = I$ and get

$$E_{32}P_{23}E_{31}E_{21}A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{32}P_{23}E_{31}E_{21}b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

We arrived at the desired upper triangular shape. It remains to use back substitution to get $c = 2$, $b = 1$ and $a = -1$.

4. Note that A is already upper triangular. Hence, we can solve the three systems $A\mathbf{x}_1 = \mathbf{e}_1$, $A\mathbf{x}_2 = \mathbf{e}_2$, and $A\mathbf{x}_3 = \mathbf{e}_3$ to find the inverse

$$A^{-1} = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix}.$$

In the first system

$$A\mathbf{x}_1 = \begin{bmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{e}_1$$

we find

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{a} \\ 0 \\ 0 \end{pmatrix}$$

by backwards substitution. In the second system

$$A\mathbf{x}_2 = \begin{bmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{e}_2$$

we find

$$\mathbf{x}_2 = \begin{pmatrix} \frac{-b}{a} \\ 1 \\ 0 \end{pmatrix}$$

by backwards substitution. Finally, we find

$$\mathbf{x}_3 = \begin{pmatrix} \frac{bd-c}{a} \\ -d \\ 1 \end{pmatrix}$$

in the third system

$$A\mathbf{x}_3 = \begin{bmatrix} a & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{e}_3$$

by backward substitution. The inverse of A is hence given by

$$A^{-1} = \begin{bmatrix} | & | & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{a} & \frac{bd-c}{a} \\ 0 & 1 & -d \\ 0 & 0 & 1 \end{bmatrix}$$

whenever $a \neq 0$. In the case where $a = 0$, the matrix A is not invertible as its columns are not linearly independent (the first column is $\mathbf{0}$). In other words, A is invertible for all choices of $a, b, c, d \in \mathbb{R}$ where $a \neq 0$.

5. **a)** The inverse of A is $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. To justify this, it suffices to check that AA^{-1} indeed equals I. But let us still explain how we found A^{-1}: Finding A^{-1} can be done by finding vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$ (columns of A^{-1}) such that

$$A\mathbf{v}_i = \mathbf{e}_i$$

for all $i \in \{1, 2, 3, 4\}$, where $\mathbf{e}_i$ is the i-th standard unit vector. Using e.g. elimination to solve these systems, we get

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and thus $A^{-1} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Alternatively, one might also be able to guess the vectors $\mathbf{v}_1, \dots, \mathbf{v}_4$ by noticing that by subtracting the $(i+1)$-th column of A from the i-th column of A, we get $\mathbf{e}_i$, for all $i \in \{1, 2, 3\}$ (and that $\mathbf{v}_4 = \mathbf{e}_4$).

b) We solve this exercise by guessing that the pattern from a) also works in general. Concretely, we define the matrix $B \in \mathbb{R}^{m \times m}$ with columns $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^m$ such that $\mathbf{v}_i := \mathbf{e}_i - \mathbf{e}_{i+1}$ for all $i \in \{1, 2, \dots, m-1\}$ and $\mathbf{v}_m := \mathbf{e}_m$. We claim that B is the unique inverse of A. To prove this, first observe that the i-th row of A is given by $\sum_{k=1}^{i} \mathbf{e}_k^\top$. This means that the entry $(AB)_{ij}$ is given by

$$(AB)_{ij} = \Big(\sum_{k=1}^{i} \mathbf{e}_k^\top\Big)\mathbf{v}_j = \sum_{k=1}^{i} \mathbf{e}_k^\top \mathbf{v}_j$$

for all $i, j \in [m]$. Let now $i, j \in [m]$ be arbitrary. We distinguish three cases.

- Assume first $j < i$. Then we get $\sum_{k=1}^{i} \mathbf{e}_k^\top \mathbf{v}_j = \sum_{k=1}^{i} \mathbf{e}_k^\top (\mathbf{e}_j - \mathbf{e}_{j+1}) = \mathbf{e}_j^\top \mathbf{e}_j - \mathbf{e}_{j+1}^\top \mathbf{e}_{j+1} = 0$.
- Next, assume $j > i$. Observe that in this case, we have $\mathbf{e}_k^\top \mathbf{v}_j = 0$ for all $k \in [i]$ and thus $(AB)_{ij} = 0$.
- Finally, we observe that $\sum_{k=1}^{i} \mathbf{e}_k^\top \mathbf{v}_j = \mathbf{e}_j^\top \mathbf{e}_j = 1$ for $i = j$.

We conclude that $AB = I$ and thus B is the unique inverse of A.

6. We want to prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent. Consider the matrices

$$W := \begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \\ | & | & | \end{bmatrix}, V := \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}, \text{ and } M := \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Observe that we have chosen M such that by definition of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, we have $W = VM$. Observe first that V has rank 3 and is invertible, since its columns are linearly independent (Inverse Theorem).

Next, we compute the rank of M. From the lecture, we know that the rank of a matrix is equal to the number of pivots after using Gauss elimination on the matrix. We use this on M: subtracting the first row of M once from its second row, we get the triangular matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which means that M has rank 3 as well. In other words, the columns of M are linearly independent and hence M is invertible.

By Lemma 2.59, we conclude that W is invertible as it can be written as the product of two invertible matrices. By the Inverse Theorem, the columns of W are linearly independent, as desired.