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Summary Lectures 21 and 22: the determinant of square matrices, see Chapter 7

2×2- matrices

The determinant of a matrix $A \in \mathbb{R}^{2\times 2}$

For $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ we define

$$det(A) = ad - bc.$$

Properties of the determinant:

Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

- $|\det(A)|$ corresponds to the volume of the parallelogram spanned by the two column vectors of A.
- For 2-by-2 matrices A, W we have $\det(AW) = \det(A)\det(W)$.
- A matrix $A \in \mathbb{R}^{2\times 2}$ is invertible if and only if $\det(A) \neq 0$.

The $n \times n$ - case

Definition (Sign of a permutation)

Given a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ of n elements, its sign $\operatorname{sgn}(\sigma)$ can be 1 or -1. The sign counts the parity of the number of pairs of elements that are out of order (sometimes called inversions) after applying the permutation. In other words,

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if} \quad |(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} \text{ st } i < j,\ \sigma(i) > \sigma(j)| \text{ even} \\ -1 & \text{if} \quad |(i,j) \in \{1, \ldots, n\} \times \{1, \ldots, n\} \text{ st } i < j,\ \sigma(i) > \sigma(j)| \text{ odd} \end{cases}$$

Definition (Π_n is the set of all permutations of n elements.)

Given $A \in \mathbb{R}^{n \times n}$, the determinant $\det(A)$ is defined as

$$\det(A) = \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}.$$

Permutations in small dimension

$n=2$: two permutations: σ_1 identity and σ_2 swaps two elements (sign -1).

$$\det(A) = (+1)\prod_{i=1}^{2} A_{i,\sigma_1(i)} + (-1)\prod_{i=1}^{2} A_{i,\sigma_2(i)} = A_{11}A_{22} - A_{12}A_{21}.$$

$n=3$

$$\begin{aligned}
\det(A) &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
&= \begin{vmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{vmatrix} + \begin{vmatrix} & A_{12} & \\ A_{21} & & \\ & & A_{33} \end{vmatrix} + \begin{vmatrix} & A_{12} & \\ & & A_{23} \\ A_{31} & & \end{vmatrix} \\
&\quad + \begin{vmatrix} & & A_{13} \\ & A_{22} & \\ A_{31} & & \end{vmatrix} + \begin{vmatrix} & & A_{13} \\ A_{21} & & \\ & A_{32} & \end{vmatrix} + \begin{vmatrix} A_{11} & & \\ & & A_{23} \\ & A_{32} & \end{vmatrix} \\
&= A_{11}A_{22}A_{33} - A_{12}A_{21}A_{33} + A_{12}A_{23}A_{31} \\
&\quad - A_{13}A_{22}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32}.
\end{aligned}$$

The key results in a nutshell

Proposition 7.2.4 - Theorem 7.2.6

- Given a matrix $A \in \mathbb{R}^{n \times n}$ we have
$$\det(A^\top) = \det(A).$$
- Given a triangular (either upper- or lower-) matrix $T \in \mathbb{R}^{n \times n}$ we have
$$\det(T) = \prod_{k=1}^{n} T_{kk}.$$
  In particular, $\det(I) = 1$.
- If $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix then $\det(Q) = \pm 1$.
- A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if
$$\det(A) \neq 0.$$
- Given matrices $A, B \in \mathbb{R}^{n \times n}$ we have $\det(AB) = \det(A)\det(B)$.
- Given an invertible matrix $A \in \mathbb{R}^{n \times n}$, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Cramer's Rule: a formula for linear systems

Example $n = 3$. Assume A is n by n and $\det(A) \neq 0$

If $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then we have

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{bmatrix}.$$

Theorem (Cramer's Rule)

Let $A \in \mathbb{R}^{n \times n}$ such that $\det(A) \neq 0$ and $b \in \mathbb{R}^n$ then the solution $x \in \mathbb{R}^n$ of $Ax = b$ is given by

$$x_j = \frac{\det(\mathscr{B}_j)}{\det(A)},$$

where $\mathscr{B}_j$ is the matrix obtained from A by replacing its j-th column by b.