Robert Weismantel

Summary Lecture 26: Symmetric matrices and the spectral theorem

Our next target: Symmetric matrices

Target

Here we consider symmetric matrices $A \in \mathbb{R}^{n \times n}$, $A^T = A$. Our target is to show that such a matrix always has a complete set of eigenvectors.

Lemma 9.2.7 and 9.2.8

- Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and λ an eigenvalue of A, then $\lambda \in \mathbb{R}$.
- Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $\lambda_1 \neq \lambda_2$ two distinct eigenvalues of A with corresponding eigenvectors v_1, v_2. Then v_1 and v_2 are orthogonal.

The Spectral Theorem

Every symmetric matrix $A \in \mathbb{R}^{n \times n}$ has n real eigenvalues and an orthonormal basis made of eigenvectors of A.

Consequences of the spectral theorem

Corollary 9.2.2

For any symmetric matrix $A \in \mathbb{R}^{n \times n}$ there exists an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ (whose columns are eigenvectors of A) such that

$$A = V \Lambda V^\top,$$

where $\Lambda \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the eigenvalues of A in its diagonal.

Let A be a real $n \times n$ symmetric matrix

Let $v_1, \ldots, v_n$ be an orthonormal basis of eigenvectors of A and $\lambda_1, \ldots, \lambda_n$ the associated eigenvalues. Then $A = \sum_{i=1}^{n} \lambda_i v_i v_i^\top$

Corollary 9.2.4

The rank of a real symmetric matrix A is the number of non-zero eigenvalues.