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Summary of Lecture 15: Orthogonal vectors, orthogonal complements of subspaces

Orthogonality of vectors and subspaces

The target

Orthogonality is a key concept that allows us to decompose a space into two subspaces, understand systems of linear equations in more detail. It even allows us to define a pseudoinverse, a topic discussed later on.

Definition 5.1.1

Vectors $v, w \in \mathbb{R}^n$ are orthogonal/ perpendicular (see Def. 1.15) if

$$v^T w = \sum_{i=1}^{n} v_i w_i = 0.$$

Subspaces V and W are orthogonal if for all $v \in V$ and $w \in W$, the vectors v and w are orthogonal.

Preliminaries

Lemma 5.1.2

Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_l$ be bases of subspace V and W. V and W are orthogonal if and only if v_i and w_j are orthogonal for all i and j.

Lemma 5.1.3

Let V and W be two orthogonal subspaces of $\mathbb{R}^n$. Let $v_1, \ldots, v_k$ be a basis of subspace V. Let $w_1, \ldots, w_l$ be a basis of subspace W. The set of vectors $\{v_1, \ldots, v_k, w_1, \ldots, w_l\}$ are linearly independent.

Corollary 5.1.4

Let V and W be orthogonal subspaces of $\mathbb{R}^n$. Then $V \cap W = \{0\}$. Moreover,

$$V + W = \{v + w \mid v \in V,\ w \in W\}$$

is a subspace of $\mathbb{R}^n$ and

$$\dim(V + W) = \dim(V) + \dim(W) \leq n.$$

The orthogonal complement of a subspace

Definition 5.1.5

Let V be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of V as

$$V^\perp = \{w \in \mathbb{R}^n \mid w^T v = 0 \text{ for all } v \in V\}.$$

$V^\perp$ is a subspace of $\mathbb{R}^n$!

A first application of this definition is given next.

Theorem 5.1.6

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Then $N(A) = C(A^T)^\perp = R(A)^\perp$.

Theorem 5.1.7

Let V, W be orthogonal subspaces of $\mathbb{R}^n$. The statements are equivalent.

(i) $W = V^\perp$.

(ii) $\dim(V) + \dim(W) = n$.

(iii) Every $u \in \mathbb{R}^n$ can be written as $u = v + w$ with unique $v \in V$, $w \in W$.