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Summary Lecture 16: A decomposition of $\mathbb{R}^n$ and projections of vectors

Orthogonality allows for a decomposition of $\mathbb{R}^n$

Lemma 5.1.8

Let V be a subspace of $\mathbb{R}^n$. Then $V = (V^\perp)^\perp$.

Notice that for any subspace V of $\mathbb{R}^n$,

$\mathbb{R}^n = V + V^\perp = \{v + w \mid v \in V,\ w \in V^\perp\}$.

In particular, we obtain

Corollary 5.1.9

For $A \in \mathbb{R}^{m \times n}$, $N(A) = C(A^T)^\perp$ and $C(A^T) = N(A)^\perp$.

A further link between the nullspaces of A and A^TA is given below.

Lemma 5.1.10

Let $A \in \mathbb{R}^{m \times n}$. Then $N(A) = N(A^TA)$ and $C(A^T) = C(A^TA)$.

Projections: a key geometric concept

Definition 5.2.1

The projection of a vector $b \in \mathbb{R}^m$ on a subspace S (of $\mathbb{R}^m$) is the point in S that is closest to b. In other words

$$\text{proj}_S(b) = \underset{p \in S}{\text{argmin}} \|b - p\|. \tag{1}$$

The one-dimensional case

Intuition: Let S be the subspace corresponding to aline that goes through the vector $a \in \mathbb{R}^m \setminus \{0\}$, i.e. $S = \{\lambda a \mid \lambda \in \mathbb{R}\} = C(a)$. The projection p is the vector in the subspace S such that the "error vector" $e = b - p$ is perpendicular to a.

Lemma 5.2.2

Let $a \in \mathbb{R}^m \setminus \{0\}$. The projection of $b \in \mathbb{R}^m$ on $S = \{\lambda a \mid \lambda \in \mathbb{R}\} = C(a)$ is

$$\text{proj}_S(b) = \frac{a^T b}{a^T a} a.$$

The general case

The idea is similiar to the one-dimensional case

Let S be a subspace in $\mathbb{R}^m$ generated by $a_1, \ldots, a_n \in S$, i.e.,

$$S = \text{ span}(a_1, \ldots, a_n) = C(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

where

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}.$$

Lemma 5.2.3

The projection of a vector $b \in \mathbb{R}^m$ to the subspace $S = C(A)$ can be written as

$$\text{proj}_S(b) = A\hat{x}, \text{ where } \hat{x} \text{ satisfies the normal equations } A^T A\hat{x} = A^T b.$$

Recall for $m = 1$

$$\text{proj}_S(b) = \lambda^* a = \frac{a^T b}{a^T a} a \iff a^T a\lambda^* a = a^T ba \iff a^T a\lambda^* = a^T b.$$