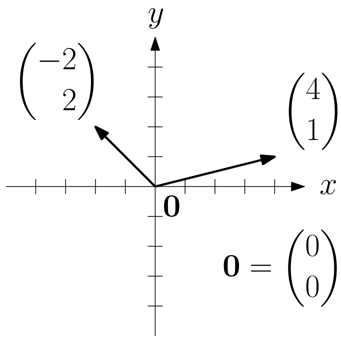
Lecture plan — Linear Algebra (401-0131-00L, HS25), ETH Zürich
Numbering of Sections, Definitions, Figures, etc. as in the Lecture Notes

Week 0

Vectors and linear combinations (Section 1.1)

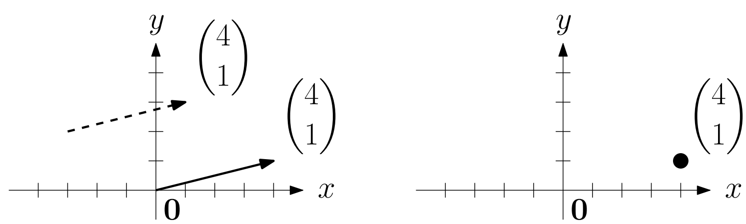
Definition 1.1: Let $m \geq 0$ be a natural number. An m-dimensional (coordinate) vector is an element of $\mathbb{R}^m$. $\qquad m \in \mathbb{N} = \{0, 1, 2 \ldots\}$ (set of natural numbers)

$\mathbb{R}^2$

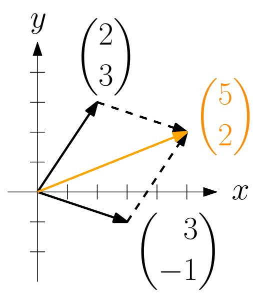


$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \qquad \mathbb{R}^m \text{ (column vectors)}$$

Drawing as arrow (movement) or point (location):



Vector addition: Combine the movements!



$$\begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$$

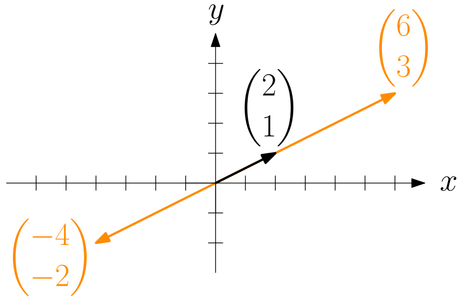
"parallelogram"

Definition 1.2: Let

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} \in \mathbb{R}^m. \text{ The vector } \mathbf{v}+\mathbf{w} := \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_m + w_m \end{pmatrix} \in \mathbb{R}^m \text{ is the sum of } \mathbf{v} \text{ and } \mathbf{w}.$$

More vectors: $\mathbf{u} + \mathbf{v} + \mathbf{w} := (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

Scalar multiplication: move λ times as far!



$$3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} \qquad (-2)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

Definition 1.3: Let

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} \in \mathbb{R}^m,\ \lambda \in \mathbb{R}. \text{ The vector } \lambda\mathbf{v} := \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_m \end{pmatrix} \in \mathbb{R}^m \text{ is a } \textit{scalar multiple} \text{ of } \mathbf{v}.$$

Linear combination: all in one!

Definition 1.4: Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$, $\lambda, \mu \in \mathbb{R}$. Then

$$\lambda\mathbf{v} + \mu\mathbf{w} \in \mathbb{R}^m$$

is a *linear combination* of $\mathbf{v}$ and $\mathbf{w}$.

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^m$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$, then

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n$$

is a *linear combination* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

$$\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}:$$

λ	μ	$\lambda\mathbf{v}$	$\mu\mathbf{w}$	$\lambda\mathbf{v} + \mu\mathbf{w}$
-3	2	$\begin{pmatrix} -6 \\ -9 \end{pmatrix}$	$\begin{pmatrix} 6 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -11 \end{pmatrix}$
1	-1	$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Fact 1.5: Every vector $\mathbf{u} \in \mathbb{R}^2$ is a linear combination of

$$\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Proof. Let

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2.$$

Goal: find $\lambda, \mu \in \mathbb{R}$ such that

$$\lambda \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Two equations in two variables λ and μ:

$$\begin{array}{rcl} 2\lambda + 3\mu & = & u_1 \\ 3\lambda - 1\mu & = & u_2. \end{array}$$

Add $3 \cdot$ (equation 2) to equation 1.

$$\begin{array}{rcrcrcr} 2\lambda & + & 3\mu & = & u_1 & & \\ 9\lambda & - & 3\mu & = & & & 3u_2 \\ \hline 11\lambda & & & = & u_1 & + & 3u_2 \end{array}$$

Solve for λ:

$$\lambda = \frac{u_1 + 3u_2}{11}.$$

Solve for μ (equation 1):

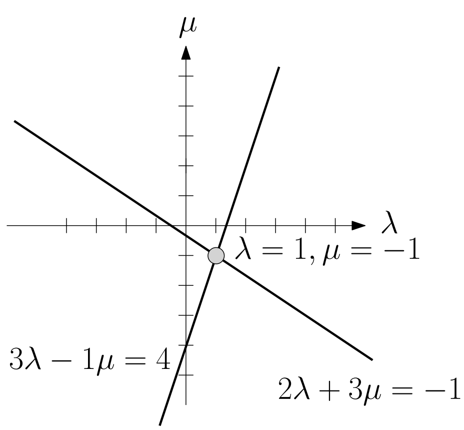
$$3\mu = u_1 - 2\lambda = u_1 - \frac{2u_1 + 6u_2}{11} = \frac{11u_1 - (2u_1 + 6u_2)}{11} = \frac{9u_1 - 6u_2}{11}.$$

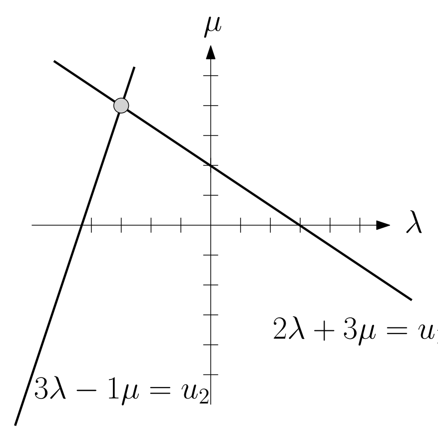
Divide by 3:

$$\mu = \frac{3u_1 - 2u_2}{11}.$$

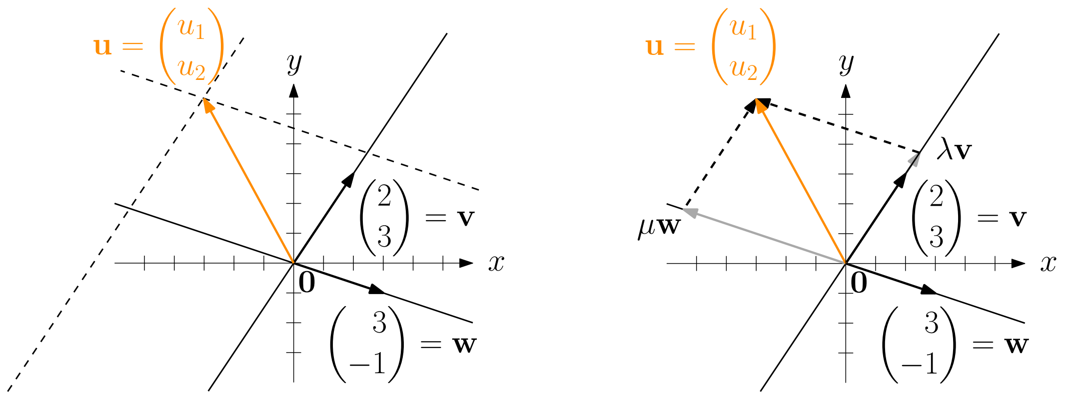
$\square$

Row picture (Figure 1.7): $\mathbf{u} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ and general $\mathbf{u}$

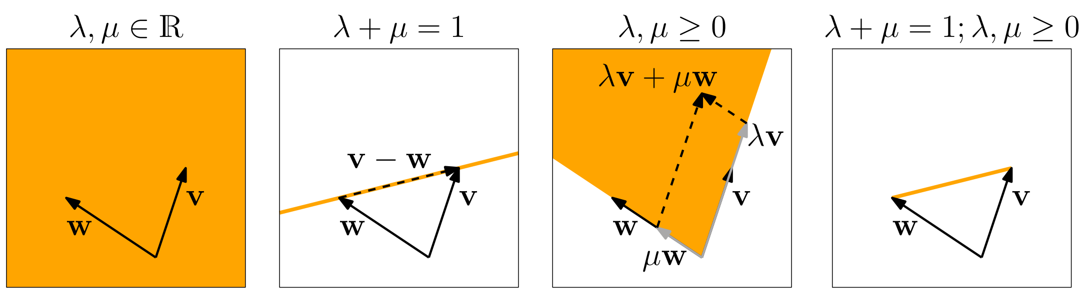




Column picture (Figure 1.8): construct the parallelogram!



Affine, conic, convex combinations: special linear combinations (Definition 1.7)



linear | (i) affine | (ii) conic | (iii) convex

$$\begin{aligned} &\lambda\mathbf{v} + \mu\mathbf{w} \\ &= \lambda\mathbf{v} + (1-\lambda)\mathbf{w} \\ &= \mathbf{w} + \lambda(\mathbf{v} - \mathbf{w}) \end{aligned}$$